Particle Scattering and Number Theory

Lance Dixon

Recent Developments in Scattering Amplitudes

Initiative for Theoretical Sciences

CUNY Graduate Center

November 5, 2021

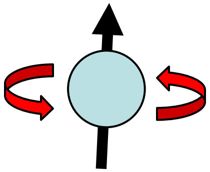
SLAC NATIONAL ACCELERATOR LABORATORY

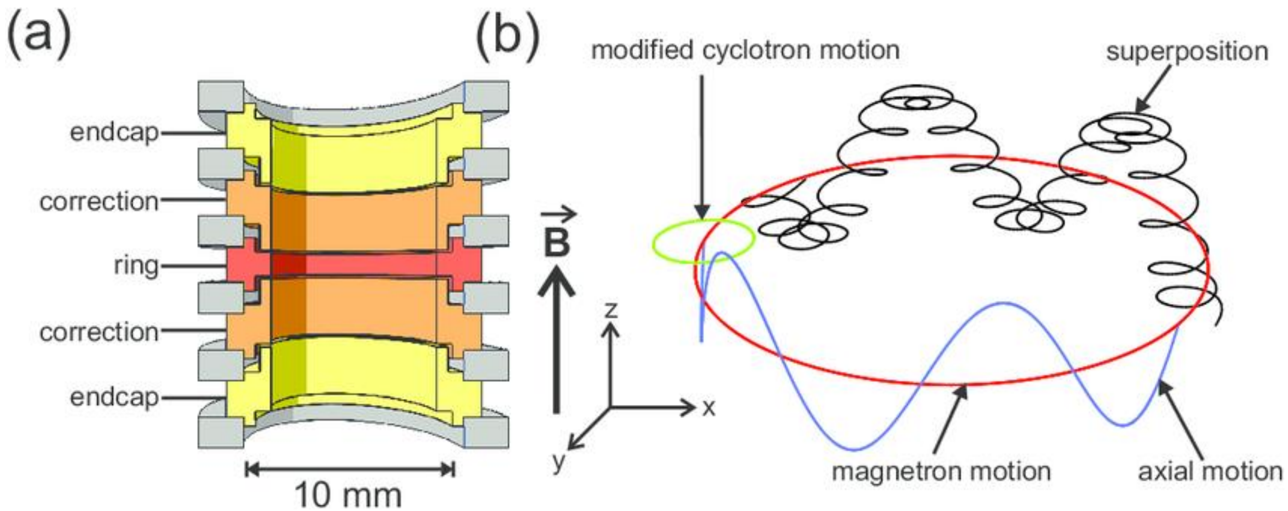
Outline

- Introduction: electron g-2 and scattering at LHC
- Numbers: Rational, irrational, transcendental
- Euler sums (MZVs) and iterated integrals
- The co-action principle
- Electron g-2 redux
- Scattering in planar N=4 SYM
- ϕ^4 theory
- Summary and outlook

The electron anomalous magnetic moment, a (precious) “baby” scattering amplitude

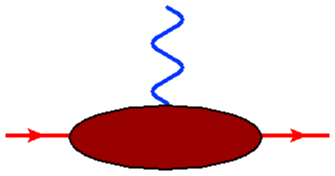
$$\vec{\mu}_e = g_e \frac{e\hbar}{2m_e c} \vec{S}_e$$





BASE, Eur. Phys. J. ST 224, 16, 3055 (2015)

Measurement doesn’t look much like particle scattering, but $a_e = (g_e - 2)/2$ can be computed from spin-flip part of $\gamma e \rightarrow e$ process as photon momentum → 0.



Shelter Island (NY), June 1947

NAS Archives

Dirac theory of electron incomplete:

- Willis Lamb (Columbia) reports on Lamb shift between 2S and 2P hydrogen
- Isadore Rabi reports on electron anomaly [Nafe, Nelson, Rabi (Columbia)]

On Quantum-Electrodynamics and the Magnetic Moment of the Electron

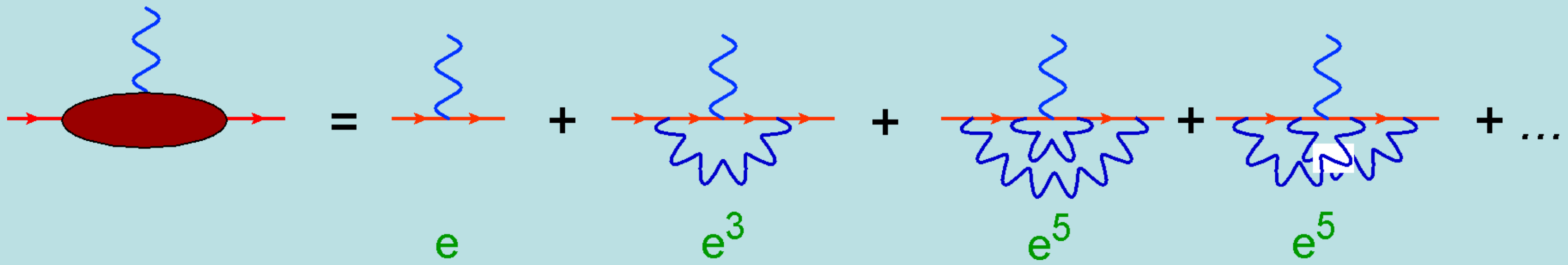
JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts
December 30, 1947

$$a_e = \frac{g-2}{2} = \frac{\alpha}{2\pi}$$

The detailed application of the theory shows that the radiative correction to the magnetic interaction energy corresponds to an additional magnetic moment associated with the electron spin, of magnitude $\delta\mu/\mu = (\frac{1}{2}\pi)e^2/\hbar c = 0.001162$. It is indeed gratifying that recently acquired experimental data confirm this prediction. Measurements on the hyperfine splitting of the ground states of atomic hydrogen and deuterium[1] have yielded values that are definitely larger than those to be expected from the directly measured nuclear moments and an electron moment of one Bohr magneton. These discrepancies can be accounted for by a small additional electron spin magnetic moment.[2]

The loop expansion

- **Feynman:** Draw all diagrams with specified incoming and outgoing particles, weight them by coupling factors at each vertex. For a given process, extra powers of coupling for each closed loop.



In quantum electrodynamics (QED), each additional loop suppressed by (Sommerfeld's) fine structure constant:

$$\frac{e^2}{4\pi\hbar c} \equiv \alpha = \frac{1}{137.035999\ldots}$$

By 3 loops, 72 Feynman diagrams

Kinoshita, Cvitanovic 1972

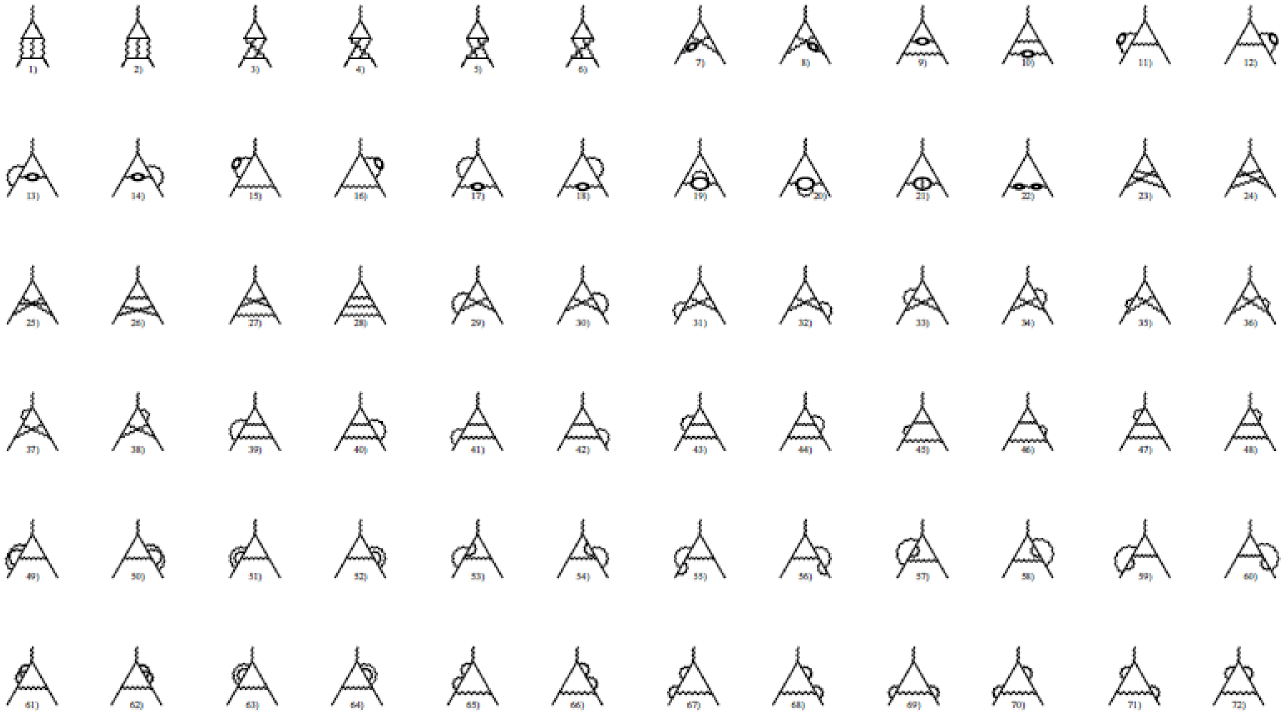


Fig. 7. The universal third order contribution to a_μ. All fermion loops here are muon–loops (first 22 diagrams). All non-universal contributions follow by replacing at least one muon in a closed loop by some other fermion

QED state of numerical art today:
5 loops, 12,672 diagrams

30 gauge invariant sets

The most difficult set,
6354 diagrams,
leading to 389 integrals.
Evaluated numerically
after Feynman
Parameterization.

Aoyama, Hayakawa, Kinoshita, Nio, Watanabe, 2006-2017

56 M. Hayakawa

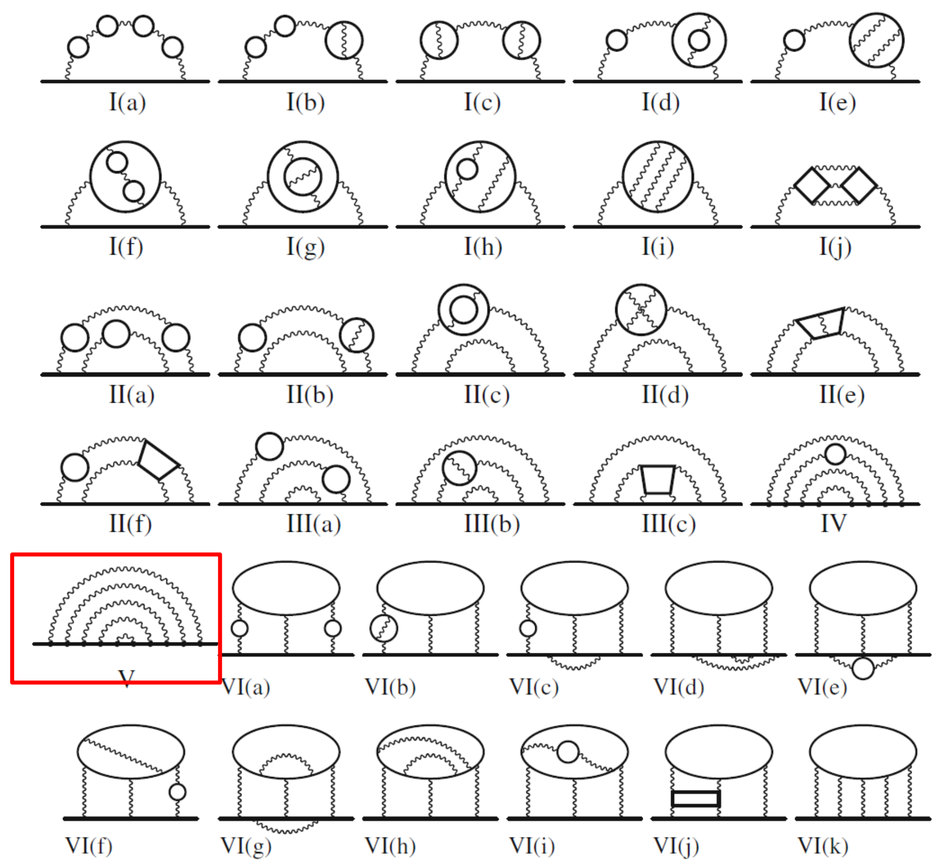


Fig. 2.7 Gauge-invariant subsets of self-energy-like diagrams at the tenth order

Seven decades of g_e-2 theory

$$a_e = \frac{\alpha}{\pi}\cdot\frac{1}{2}$$

Schwinger 1948

$$+\left(\frac{\alpha}{\pi}\right)^2\left[\frac{197}{144}+\frac{\pi^2}{12}-\frac{\pi^2}{2}\ln 2+\frac{3}{4}\zeta_3\right]$$

Karplus, Kroll 1950
Petermann 1957
Sommerfield 1957

fully analytic

$$+\left(\frac{\alpha}{\pi}\right)^3\left[\frac{28259}{5184}+\frac{17101}{810}\pi^2-\frac{298}{9}\pi^2\ln 2+\frac{139}{18}\zeta_3\right.$$
$$-\frac{239}{2160}\pi^4+\frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{1}{24}(\ln^4 2-\pi^2\ln^2 2)\right\}$$
$$\left.+\frac{83}{72}\pi^2\zeta_3-\frac{215}{24}\zeta_5\right]+\ldots$$

$$\zeta_p = \sum_{k=1}^{\infty}\frac{1}{k^p}$$
$$\mathrm{Li}_4\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty}\frac{1}{2^k k^4}$$

Kinoshita, Cvitanovic 1972
Laporta, Remiddi 1996

numerical

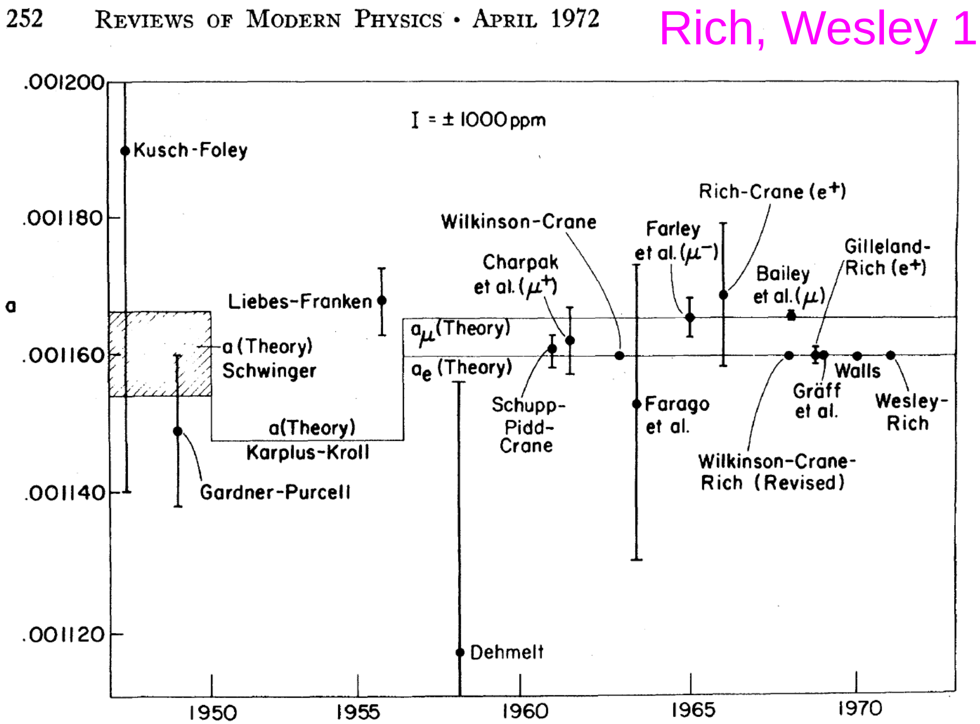
$$= 0.5\frac{\alpha}{\pi}$$
$$-0.328478965579\ldots\left(\frac{\alpha}{\pi}\right)^2$$
$$+1.181241456587\ldots\left(\frac{\alpha}{\pi}\right)^3$$
$$-1.912245764926\ldots\left(\frac{\alpha}{\pi}\right)^4$$
$$+6.7(\pm 0.2)\left(\frac{\alpha}{\pi}\right)^5$$

Aoyama, Hayakawa, Kinoshita, Nio, 2005-2007
Laporta arXiv:1704.06996

Aoyama, Hayakawa, Kinoshita, Nio, Watanabe, 2006-2017

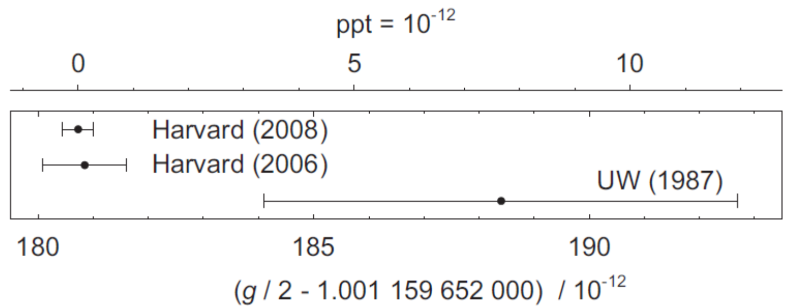
(+ mass-dep.)

Matches incredible advances in experimental precision



Rich, Wesley 1972

Van Dyck, Schwinberg, Dehmelt, 1977-1987



Hanneke, Fogwell Hoogerheide, Gabrielse, 2006-2010

$$g/2 = 1.001\ 159\ 652\ 180\ 73\ (28) \quad [0.28\ \text{ppt}]$$

Electron magnetic anomaly anomaly?

- Using new atom interferometer method, recoil measurement of mass/momentum in cesium → fine structure constant: $\alpha^{-1}(\mathrm{Cs}) = 137.035999046(27)$

Parker, Yu, Zhong, Estey, Müller, arXiv:1812.04130

- Led to 2.4σ discrepancy for electron

$$\Delta a_e \equiv a_e^{\exp} - a_e^{\rm SM} = [-87 \pm 28\,(\exp) \pm 23\,(\alpha) \pm 2\,(\text{theory})] \times 10^{-14},$$

Measuring Earth-Moon distance to width of human hair: 10^{-13}

Morel, Yao, Cladé, Guellati-Khélifa, Nature 588 (2020) 7836, 61-65

- But more recently, using rubidium

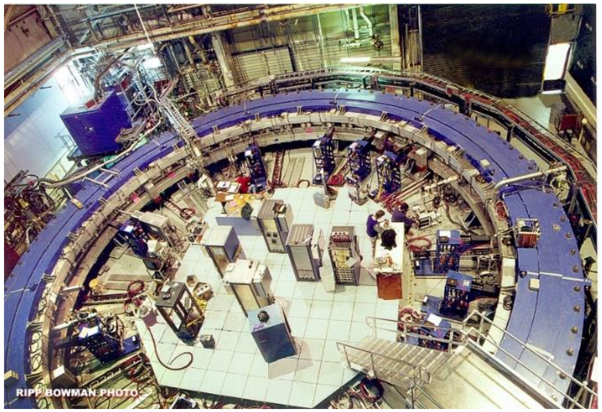
$$\alpha^{-1}(\mathrm{Rb}) = 137.035999206(11)$$

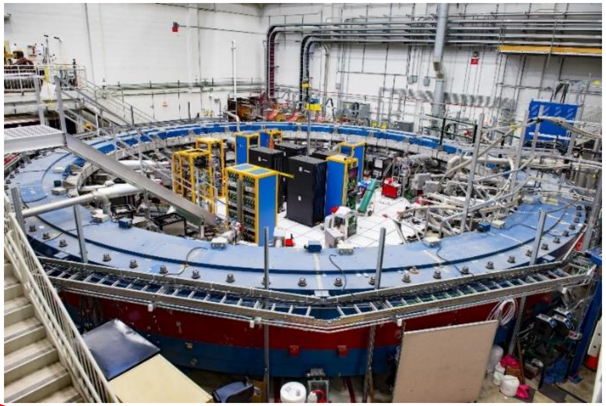
gives $\Delta a_e = [+48 \pm 30\,(\exp) \pm 2(\text{theory})] \times 10^{-14}$

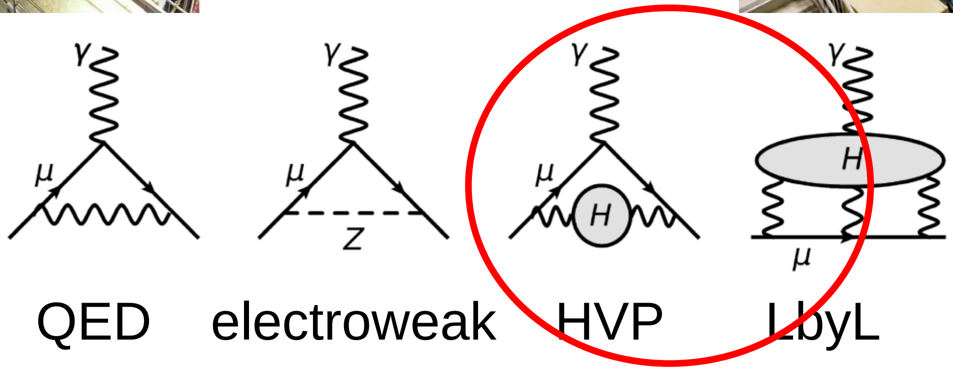
- Or 1.6σ discrepancy but the other sign!
- Wait and see!

Muon magnetic anomaly anomaly?

- Much more longstanding than electron.
- New experimental measurement from Fermilab, using muon storage ring shipped from Brookhaven.

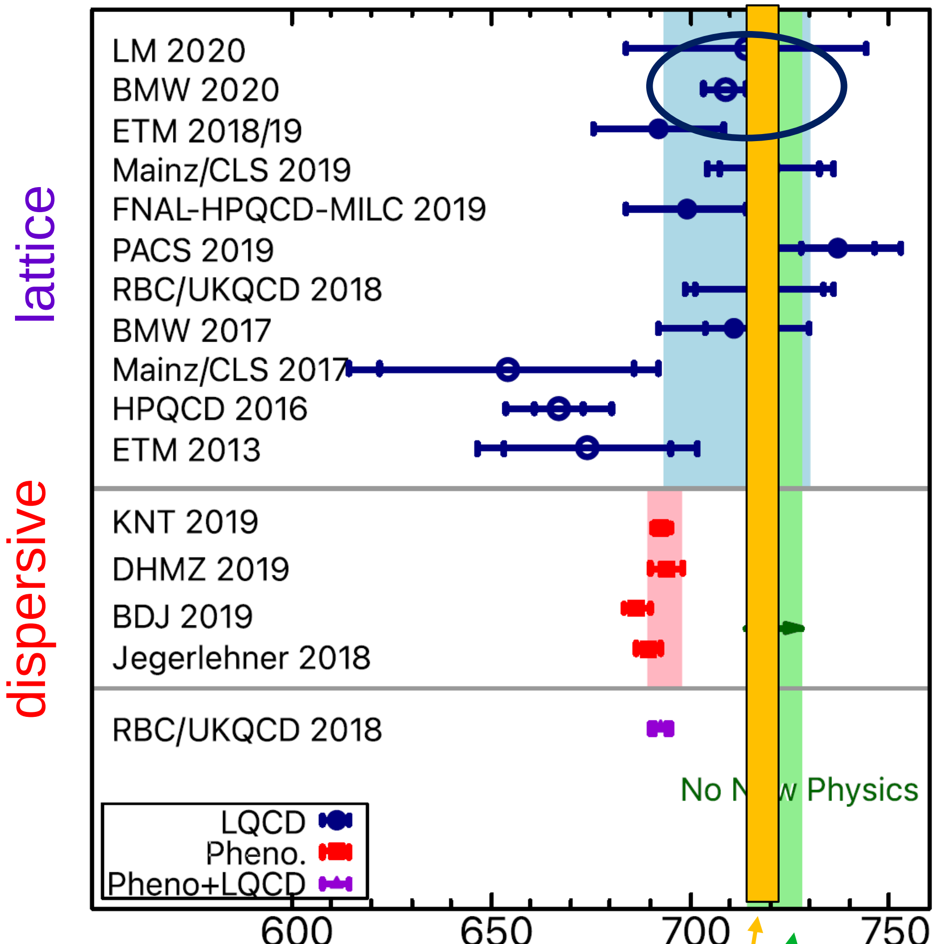






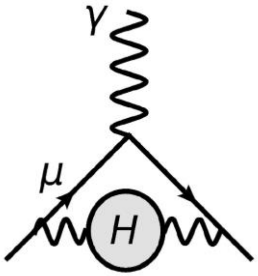
dominant theory uncertainty is not QED but QCD

Phrased as HVP "measurement"



new Fermilab g-2 result

old BNL g-2 result



Measuring Earth-Moon distance to ~ width of a human: 10^{-9}

muon g-2
theory initiative
white paper

Is discrepancy new physics or issue with evaluating HVP?

Wait and see!

Invitation:
What numbers appear (or don't) in g_e-2?

- $a_e^{(1)} = \frac{1}{2}$
- $a_e^{(2)} = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3)$
- $a_e^{(3)} = \frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2\ln 2 + \frac{139}{18}\zeta(3) - \frac{239}{2160}\pi^4 +$
  $+ \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}(\ln^4 2 - \pi^2\ln^2 2)\right\} + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5)$
- Assign "transcendental weight" w ("number of integrations") to numbers in the formulas:

$$w[\pi] = w[\ln(x)] = 1,$$

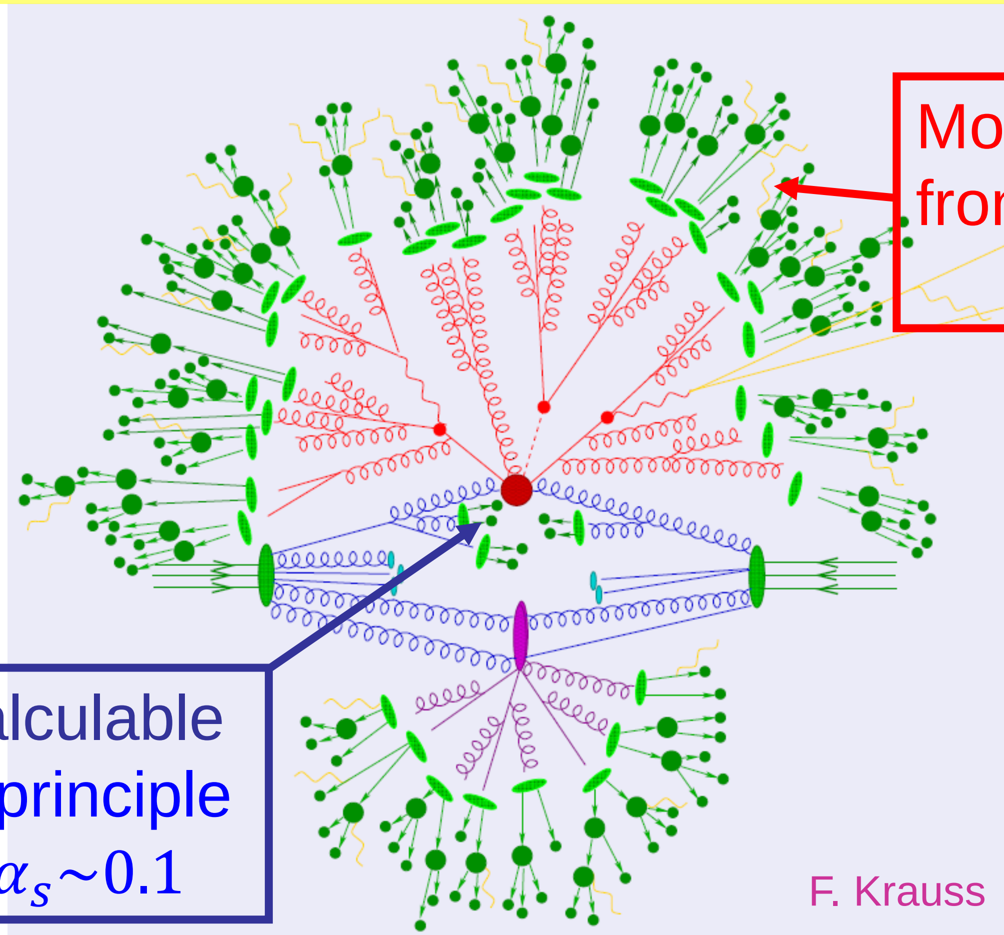
$$w[\zeta(n)] = w[\mathrm{Li}_n(x)] = n$$

- Apparently $w \leq 2L - 1$ ($L =$ loop order), but some terms are missing: e.g. no $\ln 2$, $\ln^2 2$ or $\ln^3 2$ in $a_e^{(2)}$
- Do missing terms at lower loops imply missing terms at higher loops? **YES**, once understood how to write them
- Do such patterns appear in other contexts? **YES**

Large Hadron Collider

Scattering at LHC dominated by QCD

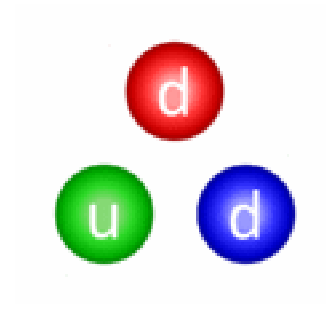
- Energies enormous, many kinematic variables

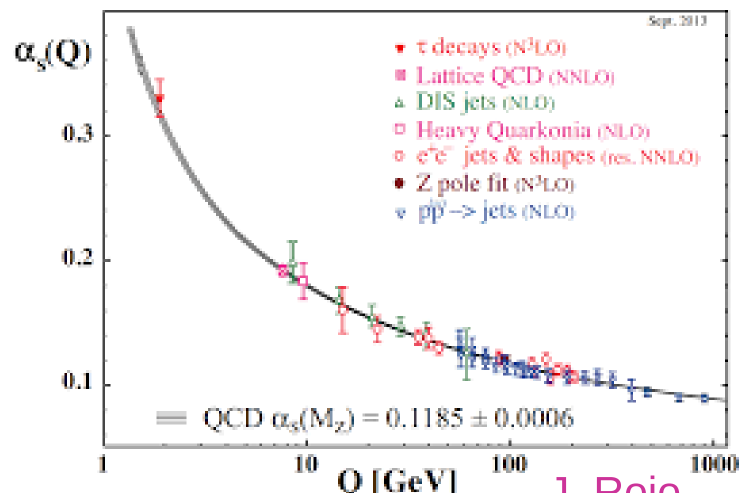


Model or get from experiment $\alpha_s \sim 1$

Calculable in principle $\alpha_s \sim 0.1$

F. Krauss

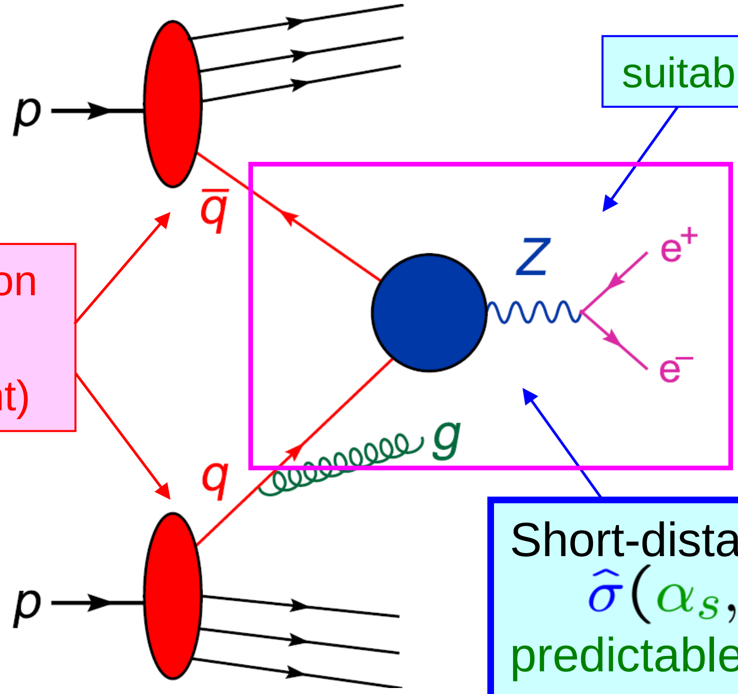




J. Rojo

QCD Factorization at LHC

At short distances, quarks and gluons (partons) in proton are almost free.
Sampled "one at a time" before exchange of binding gluons



Parton distribution functions (from experiment)

suitable final state

The "femto-universe"

size = factorization scale μ_F ("arbitrary")

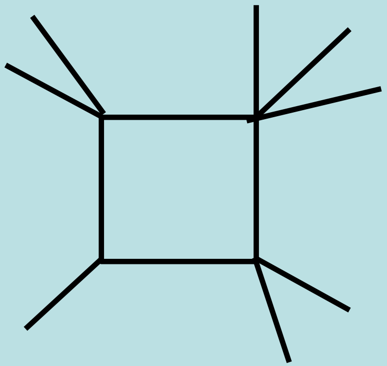
Short-distance cross section
$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R)$$
predictable using perturbative QCD
Need multi-loop amplitudes for high precision.
May depend on many kinematic variables.

One loop amplitudes

- Numbers are very simple.
- At one loop all integrals are reducible to scalar box integrals + simpler

→ combinations of dilogarithms

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

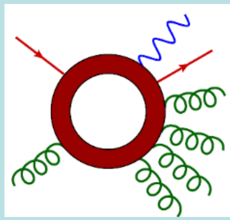
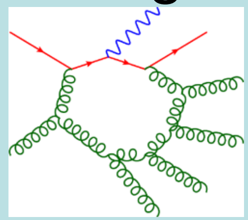


+ logarithms and rational terms

‘t Hooft, Veltman (1974)

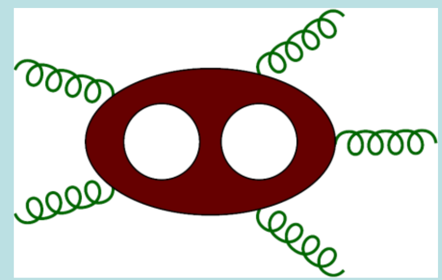
Two loops much harder

- One-loop QCD amplitudes with up to 8 external legs computed efficiently, for e.g. 2 → 6 process   pp → W + 5 jets

 =  + 256,264 more

Bern, LD, et al., 1304.1253

- Two-loop (NNLO) QCD frontier: 2 → 3 all massless scattering in large N_c (planar) limit



Badger et al., 1712.02229, 1811.11669;
Abreu et al., 1712.03946, 1812.04586, 1904.00945

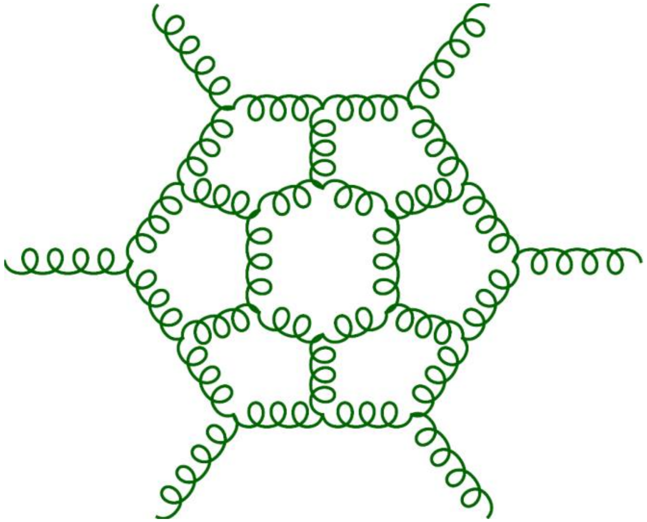
or one massive leg

Abreu et al., 2005.04195; Badger et al., 2107.14733;

- Two-loop integrals are intricate, transcendental, multi-variate functions. Special values ~ those found in g_e-2

Number-theory patterns in real scattering?

- Some patterns visible in QCD
- However, we can see them most easily in a "toy theory", planar N=4 SYM, whose remarkable symmetries let us compute 6-point amplitudes up to 7 loops!



Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116

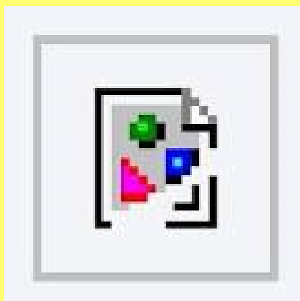
A brief history of numbers

- In the beginning, there were the integers $\mathbb{Z}$
- Then linear equations had to be solved → rational numbers $\mathbb{Q}$
- Solve polynomial equations over $\mathbb{Q}$ → algebraic numbers $\mathbb{A}$, e.g. $\sqrt{2}$, $\varphi = \frac{1+\sqrt{5}}{2}$
- $\mathbb{A}$ still far from enough to express many physical formulas

Transcendental numbers

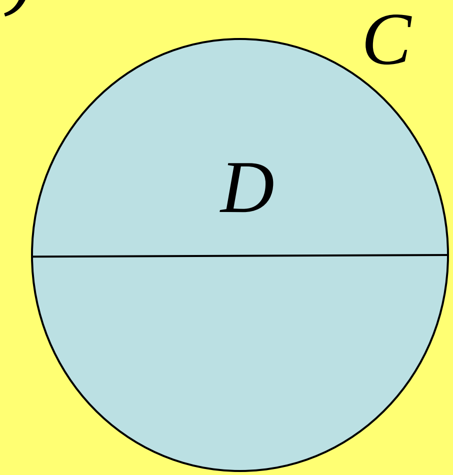
- $\pi = \frac{C}{D} = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$

Madhava-Leibniz series



1300's

1676



- Special value of a special function:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Leonhard Euler

1707-1783

See I. Todorov, 1804.09553

- ~1726: Euler wins prize essay on ship-building, although he had never been on a ship before.
- Offer to join St. Petersburg Academy,
commissioned into Russian navy (not for long).
- In 1729, Euler began to play with values of infinite series.
- In particular, the “Basel problem”:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = ???$$

Euler sums

- Euler considered also the more general quantities, now called Riemann zeta values,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

- Numerical convergence poor, important given computational tools of the day

- Euler realized that for faster convergence, one should embed $\zeta(n)$ into the alternating sums,

$$\phi(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^n} = (1 - 2^{1-n})\, \zeta(n)$$

Euler and the dilogarithm

- Euler also recognized $\zeta(2)$ and $\phi(2)$ as special values of a function, an iterated integral now called the dilogarithm [Leibniz → J. Bernoulli → Euler]:

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x \frac{dt}{t} \int_0^t \frac{dt'}{1-t'}$$

$$\mathrm{Li}_2(1) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2),$$

$$\mathrm{Li}_2(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\phi(2)$$

Functional equation for better convergence

- Differentiating dilogarithm, $\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{\ln(1-x)}{x}$

gives Euler's functional equation:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln x \ln(1-x) = \mathrm{Li}_2(1)$$

- Setting $x = \frac{1}{2}$ to be well inside radius of convergence 1, Euler could get "high precision numerics", and found $\zeta(2) = \frac{\pi^2}{6}$, and later $\zeta(2n) = -\frac{B_{2n}}{2(2n)!}(2\pi i)^n$
- But $\zeta(3) = ???$

- "For n odd all my efforts have been useless until now" [Euler, 1749]

Is $\zeta(3)$ Transcendental?

- Still not known!
- $\zeta(3)$ is proven to be irrational Apéry, 1973
- Also proven: For any $\varepsilon > 0$, at least $2^{(1-\varepsilon)\frac{\ln s}{\ln \ln s}}$ of the odd Riemann ζ values between 3 and s are irrational.
  Fischler, Sprang, Zudilin, 1803.08905
- It is a "folklore conjecture" (i.e. all physicists believe it) that $\pi, \zeta(3), \zeta(5)$, ... are algebraically independent over $\mathbb{Q}$
- Follows from Grothendieck's period conjecture for mixed Tate motives, but this seems impossible to prove
- To make formal mathematical progress, usually define motivic multiple zeta values, $\zeta \to \zeta^{\mathfrak{M}}$
- We won't worry about the distinction here.

Euler's useless efforts not so useless

- While failing to find polynomial relations among $\zeta(n)$, Euler introduced nested sums, or multiple zeta values (MZV's):

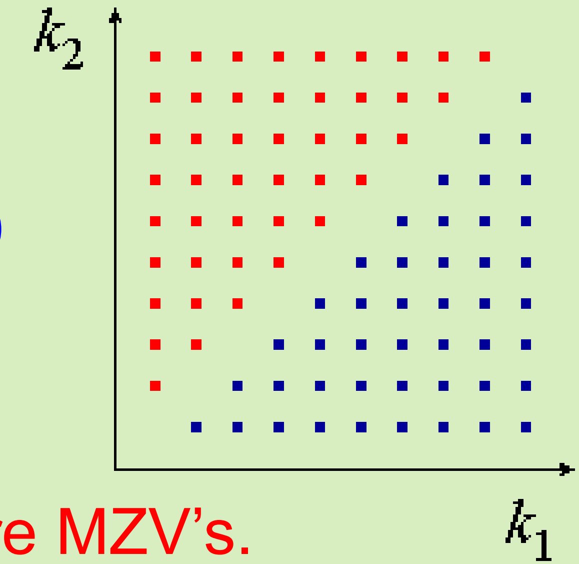
$$\zeta(n_1, \ldots, n_d) = \sum_{k_1 > \cdots > k_d > 0} \frac{1}{{k_1}^{n_1} \ldots {k_d}^{n_d}}$$

- Weight = $n_1 + \cdots + n_d$, depth = d
- And similar alternating [Euler-Zagier] sums with minus signs in the numerator

MZVs obey many identities

- For example, $\zeta(n_1, n_2) = \sum_{k_1>k_2>0} \frac{1}{{k_1}^{n_1} {k_2}^{n_2}}$

obeys the "stuffle" identity,



$\zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2)$

- The first irreducible MZV, that cannot be written in terms of $\zeta(n) \equiv \zeta_n$, is at weight 8, $\zeta(5,3) \equiv \zeta_{5,3}$. → High loops needed to explore MZV's.
- "MZV datamine", Blümlein, Broadhurst, Vermaseren, 0907.2557 solves all known relations to weight 24, also alternating (Euler) sums to at least weight 12

MZVs and Harmonic Polylogarithms (HPLs)

Remiddi, Vermaseren, hep-ph/9905237

- Classical polylogs $\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$

evaluate to Riemann zeta values $\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta_n$

- Define HPLs $H_{\vec{w}}(x),\ w_i \in \{0,1\}$ by iterated integration:

$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Then $$H_{n_1,\ldots,n_d}(1) \equiv H_{\underbrace{0,\ldots,0,1}_{n_1},\ldots,\underbrace{0,\ldots,0,1}_{n_d}}(1) = \zeta_{n_1,\ldots,n_d}$$

- Weight ***n*** = length of binary string; 2^n HPLs at weight ***n***
- Derivatives of just two types:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x)\ d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

HPLs and massless 2→2 scattering

$s + t + u = 0$ → one dimensionless variable, $x = -\frac{t}{s}$

- Only interesting limits are

$$s \to 0, \qquad t \to 0, \qquad u \to 0$$

$$\rightarrow x \to \infty, \qquad x \to 0, \qquad x \to 1$$

- Match singular points of HPLs $H_{\vec{w}}(x)$.
- HPLs $H_{\vec{w}}(x)$ with weight ≤ 4 describe all QCD amplitudes through 2 loops

Anastasiou, Glover, Oleari, Tejeda-Yeomans; Bern, LD, de Freitas (~2000)

- weight ≤ 6 for QCD amplitudes through 3 loops

  Henn, Mistlberger, Smirnov, Wasser, 2002;.09492

Generic iterated integrals

Chen; Goncharov; Brown

- Generalized polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions.
- Define by $G(a_1, a_2, \ldots, a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t)$
- Important property of space $\mathcal{G}$ of such functions: Hopf co-algebra Δ maps functions to products of "functions": $\Delta\mathcal{G} \subseteq \mathcal{G} \otimes \mathcal{G}'$

  Goncharov, math/0208144; Brown, 1102.1312
- Δ basically arises from chopping iterated integration contours into pieces.
- Weight is preserved, so $\Delta = \sum_{p,q=1}^{\infty} \Delta_{p,q}$ where $\Delta_{n-q,q} f^{(n)} = \sum_k f^{k,(n-q)} \otimes g^{k,(q)}$

Iterated integrals (cont.)

- Co-action $\Delta_{n-q,q} f^{(n)} = \sum_k f^{k,(n-q)} \otimes g^{k,(q)}$
- Special case $q = 1$ is just the derivative:

$$\Delta_{n-1,1} f = \sum_{s_k \in \mathcal{S}} f^{s_k} \otimes \ln s_k(x_a)$$

is equivalent to $$\frac{\partial f}{\partial x_a} = \sum_{s_k \in \mathcal{S}} f^{s_k} \frac{\partial \ln s_k}{\partial x_a}$$

- $\mathcal{S}$ = finite set of rational expressions, "symbol letters" s_k, depending on coordinates x_a
- f^{s_k} are pure functions, weight n-1
- Iterate the $\{n\text{-}1,1\}$ coproduct n times:

→ **Symbol** = {1,1,…,1} component of Δ

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Symbols and co-actions

- Symbol trivializes all complicated polylogarithmic identities
- → incredibly useful for simplifying massively complicated expressions for two-loop QCD amplitudes Duhr, 1203.0454
- However, differentiating n times loses all information about constants, MZVs, etc.
- Other components of Δ are more useful for diagnosing the structure of numbers like MZVs Brown, 1102.1310
- ∃ map between MZV's and non-abelian "f alphabet" $f_3, f_5, f_7, \ldots$ which makes the action of Δ manifest.
  $\zeta(2i+1) \to f_{2i+1}, \quad \zeta(5,3) \to -5 f_5 f_3 \equiv -5\, f_{5,3}$
- Similar alphabet for alternating sums, adding $f_1 \sim \ln 2$

Back to g_e-2

- What do two- and three-loop terms look like in f alphabet?
- O. Schnetz, 1711.05118, HyperlogProcedures MAPLE program
- $\frac{197}{144} + \frac{\zeta_2}{2} + 3\zeta_2 f_1 - f_3$
- $\frac{28259}{5184} + \frac{17101}{135}\zeta_2 + \frac{596}{3}\zeta_2 f_1 - \frac{278}{27} f_3 + \frac{511}{24}\zeta_4$
  $- \frac{350}{9} f_{1,3} - \frac{83}{9}\zeta_2 f_3 + \frac{86}{9} f_5$
- Δ contains an operation: "clip a f_{2i+1} from the left".
- Always land on something seen at lower loops.
- Conversely: no naked f_1 at two loops
  → no f_1, $f_{1,1}$, $f_{1,1,1}$, $f_{3,1}$, … expected at higher loops.

Co-action principle

Schnetz, 1302.6445; Brown, 1512.06409; Panzer, Schnetz, 1603.04289;…

- Suppose $\mathcal{H} \subset \mathcal{G}$ is some subspace of a space of generalized polylogs or MZVs which is picked out by "physics" in some way.
- Then the left factor in the co-action should be stable, i.e.

$$\Delta\mathcal{H} \subset \mathcal{H} \otimes \mathcal{K}$$

- **Note**: left ←→right here, versus f alphabet ordering
- This principle makes many predictions which can be tested in a variety of multi-loop settings.

Cosmic Galois Group

- There is a group action C dual to Δ
- The restriction $\Delta\mathcal{H} \subset \mathcal{H} \otimes \mathcal{K}$ corresponds to invariance under the group, $C \times \mathcal{H} \to \mathcal{H}$
- Group C is infinite dimensional analog of Galois group associated with roots of a polynomial equation
- Because this property appears "everywhere", termed "cosmic Galois group"

  Cartier (1996,2000); Andre (2008); Brown, 1512.06409, 1512.06410
- Precisely how group acts (what numbers appear) depends on the physical problem

g_e-2 at four loops

- Computed "almost" analytically
  Laporta arXiv:1704.06996
- Contains non-polylog terms.
  Also, polylog terms require two different f alphabets, one associated with $G(a_1, \ldots, a_n; 1)$ where a_i are 4th roots of unity, f_i^4 another with 6th roots, f_i^6 + g_1^6
- **Co-action principle satisfied:** Clipping an f_i from left lands on a stable subspace, called the Galois conjugates.

$$
\begin{aligned}
a_e \cong\ & \frac{1}{2}\left(\frac{\alpha}{\pi}\right) \\
&+ \left(\frac{197}{144} + \frac{1}{12}\pi^2 + \frac{27}{32}f_3^6 - \frac{1}{4}g_1^6\pi^2\right)\left(\frac{\alpha}{\pi}\right)^2 \\
&+ \left(\frac{28259}{5184} + \frac{17101}{810}\pi^2 + \frac{139}{16}f_3^6 - \frac{149}{9}g_1^6\pi^2 - \frac{525}{32}g_1^6 f_3^6 + \frac{1969}{8640}\pi^4 - \frac{1161}{128}f_5^6\right. \\
&\quad \left. + \frac{83}{64}f_3^6\pi^2\right)\left(\frac{\alpha}{\pi}\right)^3 \\
&+ \left(\frac{1243127611}{130636800} + \frac{30180451}{155520}\pi^2 - \frac{255842141}{2419200}f_3^6 - \frac{8873}{36}g_1^6\pi^2 + \frac{126909}{2560}\frac{f_4^6}{\mathrm{i}\sqrt{3}}\right. \\
&\quad - \frac{84679}{1280}g_1^6 f_3^6 + \frac{169703}{3840}\frac{f_2^6\pi^2}{\mathrm{i}\sqrt{3}} + \frac{779}{108}g_1^6 g_1^6\pi^2 + \frac{112537679}{3110400}\pi^4 - \frac{2284263}{25600}f_5^6 \\
&\quad + \frac{8449}{96}g_1^6 g_1^6 f_3^6 - \frac{12720907}{345600}f_3^6\pi^2 - \frac{231919}{97200}g_1^6\pi^4 + \frac{150371}{256}\frac{f_6^6}{\mathrm{i}\sqrt{3}} + \frac{313131}{1280}g_1^6 f_5^6 \\
&\quad - \frac{121383}{1280}f_2^6 f_4^6 - \frac{14662107}{51200}f_3^6 f_3^6 + \frac{8645}{128}\frac{f_2^6 g_1^6 f_3^6}{\mathrm{i}\sqrt{3}} - \frac{231}{4}g_1^6 g_1^6 g_1^6 f_3^6 - \frac{16025}{48}\frac{f_4^6\pi^2}{\mathrm{i}\sqrt{3}} \\
&\quad + \frac{4403}{384}g_1^6 f_3^6\pi^2 - \frac{136781}{1920}f_2^6 f_2^6\pi^2 + \frac{7069}{75}f_2^4 f_2^4\pi^2 - \frac{1061123}{14400}f_3^6 g_1^6\pi^2 \\
&\quad + \frac{1115}{72}\frac{f_2^6 g_1^6 g_1^6\pi^2}{\mathrm{i}\sqrt{3}} + \frac{781181}{20736}\frac{f_2^6\pi^4}{\mathrm{i}\sqrt{3}} - \frac{4049}{1080}g_1^6 g_1^6\pi^4 + \frac{90514741}{54432000}\pi^6 \\
&\quad - \frac{95624828289}{2050048}f_7^6 - \frac{29295}{512}g_1^6 f_2^6 f_4^6 + \frac{107919}{512}g_1^6 f_3^6 f_3^6 + \frac{337365}{256}f_3^6 g_1^6 f_3^6 \\
&\quad - \frac{55618247}{409600}f_5^6\pi^2 - \frac{1055}{256}g_1^6 f_2^6 f_2^6\pi^2 + \frac{26}{3}f_1^4 f_2^4 f_2^4\pi^2 + \frac{553}{4}g_1^6 f_3^6 g_1^6\pi^2 \\
&\quad - \frac{35189}{1024}f_3^6 g_1^6 g_1^6\pi^2 + \frac{79147091}{2211840}f_3^6\pi^4 - \frac{3678803}{4354560}g_1^6\pi^6 \\
&\quad \left. + \sqrt{3}(E_{4a} + E_{5a} + E_{6a} + E_{7a}) + E_{6b} + E_{7b} + U\right)\left(\frac{\alpha}{\pi}\right)^4.
\end{aligned}
$$

"Galois conjugates" through weight 5

wt.	dim.	words					
0	1	1					
1	0	—					
2	1	π^2					
3	2	f_3^6	$g_1^6\pi^2$				
4	6	f_4^6	$g_1^6 f_3^6$	$f_2^6\pi^2$	$f_2^4\pi^2$	$g_1^6 g_1^6\pi^2$	π^4
5	4	f_5^6	$g_1^6 g_1^6 f_3^6$	$f_3^6\pi^2$	$g_1^6\pi^4$		

- Weights 1 to 4 "expected to be stable"
- Weight 5 will undoubtedly have additions once next loop order is computed…

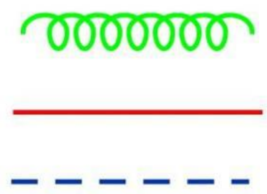
N=4 SYM particle content

Brink, Schwarz, Scherk; Gliozzi, Scherk, Olive (1977)

massless spin 1 gluon

4 massless spin 1/2 gluinos

6 massless spin 0 scalars



Gauge group:
$G = SU(N_c)$,
$N_c \to \infty$

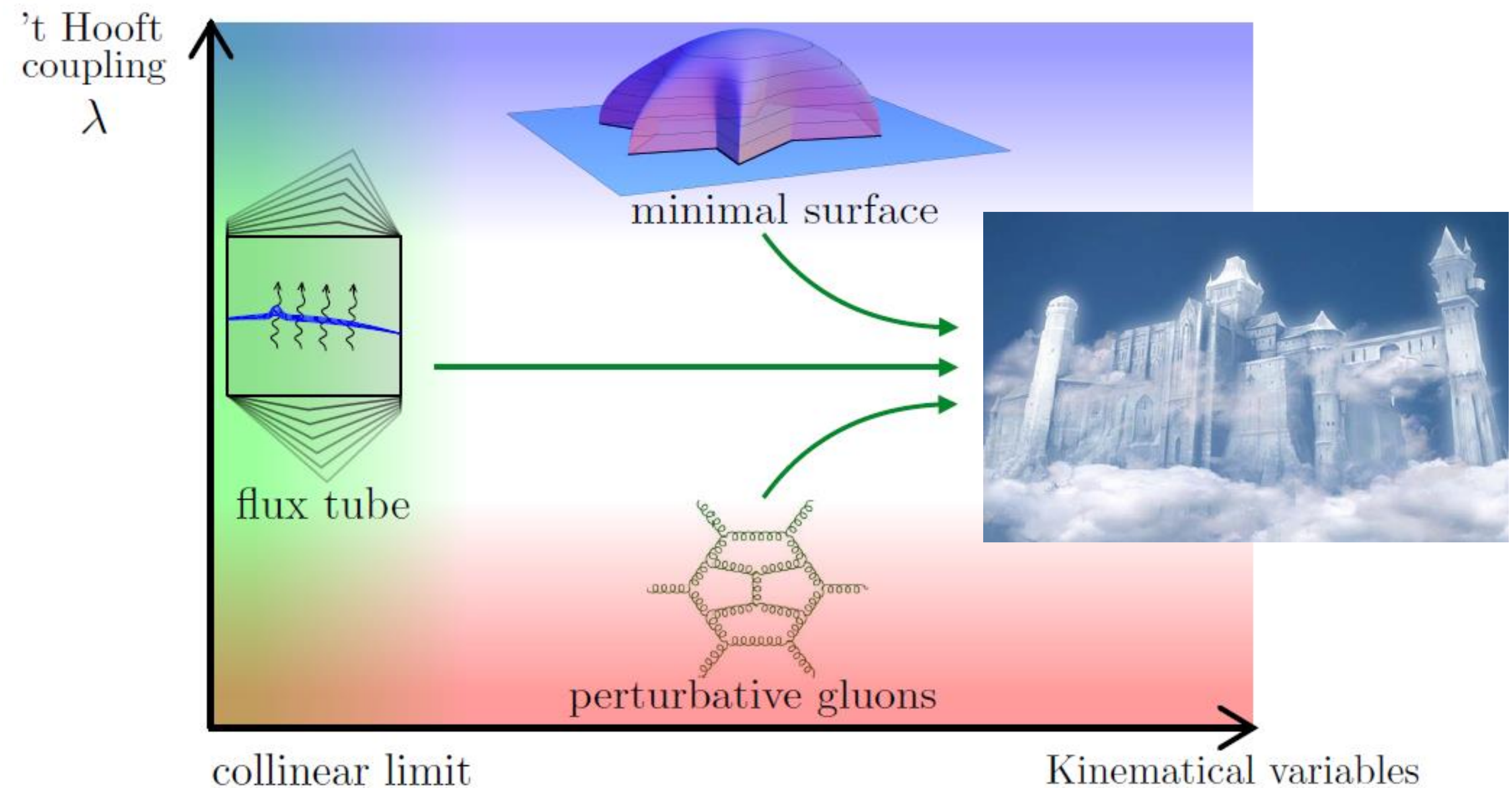
SUSY
Q_a, a=1,2,3,4
shifts helicity
by 1/2 $\longleftrightarrow$

$\mathcal{N}=4$	1 $\longleftrightarrow$	4 $\longleftrightarrow$	6 $\longleftrightarrow$	4 $\longleftrightarrow$	1
	g^-	$\lambda^-_{\bar{i}}$	$\bar{\phi}_{\bar{i}\bar{j}}, \phi_{ij}$	λ^+_i	g^+
helicity	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1

all in adjoint representation of G

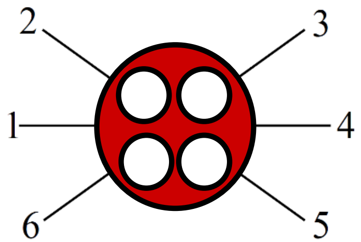
Solving for Planar N=4 SYM Amplitudes

Images: A. Sever, N. Arkani-Hamed



Bootstrapping amplitudes through 7 loops

S. Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890 and 1906.07116



- Six-gluon amplitude is first one not fixed by symmetries, depends on u, v, w (dual conformal cross ratios).
- Amplitude lives in remarkably small space of polylogarithmic **hexagon functions**, the weight $2L$ part at L loops.
- Space small enough that one can **bootstrap** the amplitude by writing a linear combination of functions and imposing constraints → unique solution.
- At $u = v = w = 1,$ the amplitudes, and all of their iterated $\{n\text{-}q,1,\ldots,1\}$ coproducts (derivatives) evaluate to MZVs.

f basis for $\mathcal{H}^{\text{hex}}(1,1,1)$

#MZV	#	basis elements / Galois conjugates
12	6	$\zeta_{12}\,,\ \ 7f_{3,9}-6\zeta_4 f_{3,5}\,,\ \ 5f_{3,9}-3\zeta_6 f_{3,3}\,,\ \ \zeta_2 f_{3,7}-\zeta_6 f_{3,3}\,,\ \ 7f_{5,7}-\zeta_2 f_{5,5}-3\zeta_4 f_{5,3}\,,\ \ 5f_{7,5}-2\zeta_2 f_{7,3}$
9	5	$33f_{11}-20\zeta_8 f_3\,,\ \ \zeta_2 f_9-\zeta_8 f_3\,,\ \ 3\zeta_4 f_7-2\zeta_8 f_3\,,\ \ 3\zeta_6 f_5-2\zeta_8 f_3\,, 5f_{3,3,5}-2\zeta_2 f_{3,3,3}+\frac{5611}{132}\zeta_8 f_3$
7	3	$\zeta_{10}\,,\ \ 7f_{3,7}-\zeta_2 f_{3,5}-3\zeta_4 f_{3,3}\,,\ \ 5f_{5,5}-2\zeta_2 f_{5,3}$
5	3	$7f_9-6\zeta_4 f_5\,,\ \ 5f_9-3\zeta_6 f_3,\ \ \zeta_2 f_7-\zeta_6 f_3$
4	2	$\zeta_8\,,\ \ \zeta_{5,3}+5\zeta_3\zeta_5-\zeta_2(\zeta_3)^2=5f_{3,5}-2\zeta_2 f_{3,3}$
3	1	$7\zeta_7-\zeta_2\zeta_5-3\zeta_4\zeta_3=7f_7-\zeta_2 f_5-3\zeta_4 f_3$
2	1	ζ_6
2	1	$5\zeta_5-2\zeta_2\zeta_3=5f_5-2\zeta_2 f_3$
1	1	ζ_4
1	0	–
1	1	ζ_2
0	0	–
1	1	1

∂_3 ∂_5

The values of the MHV amplitudes $\mathcal{E}^{(L)}(1,1,1)$ for $L = 1$ to 7 in the f-basis are:

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[5\,f_{3,5} - 2\zeta_2 f_{3,3}\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 384\left[7f_{3,7} - \zeta_2 f_{3,5} - 3\zeta_4 f_{3,3}\right] - 312\left[5\,f_{5,5} - 2\zeta_2 f_{5,3}\right],$$
$$\begin{aligned}\mathcal{E}^{(6)}(1,1,1) = &-\frac{2273108143}{6219}\zeta_{12} + 2264\left[7f_{3,9} - 6\zeta_4 f_{3,5}\right] + 6536\left[5\,f_{3,9} - 3\zeta_6 f_{3,3}\right]\\ &- 3072\left[\zeta_2 f_{3,7} - \zeta_6 f_{3,3}\right] + 5328\left[7f_{5,7} - \zeta_2 f_{5,5} - 3\zeta_4 f_{5,3}\right]\\ &+ 4224\left[5\,f_{7,5} - 2\zeta_2 f_{7,3}\right],\end{aligned}$$

$$\mathcal{E}^{(7)}(1,1,1) = \frac{2519177639}{1260}\zeta_{14} - 63968\Big[5f_{9,5} - 2\zeta_2 f_{9,3}\Big] - 77952\Big[7f_{7,7} - \zeta_2 f_{7,5} - 3\zeta_4 f_{7,3}\Big]$$
$$- 34976\Big[7f_{5,9} - 6\zeta_4 f_{5,5}\Big] - 95552\Big[5f_{5,9} - 3\zeta_6 f_{5,3}\Big] + 44640\Big[\zeta_2 f_{5,7} - \zeta_6 f_{5,3}\Big]$$
$$- \frac{413920}{11}\Big[33f_{3,11} - 20\zeta_8 f_{3,3}\Big] + 28000\Big[\zeta_2 f_{3,9} - \zeta_8 f_{3,3}\Big]$$
$$+ 62720\Big[3\zeta_4 f_{3,7} - 2\zeta_8 f_{3,3}\Big] + \frac{218696}{3}\Big[3\zeta_6 f_{3,5} - 2\zeta_8 f_{3,3}\Big]$$
$$- 4992\Big[5f_{3,3,3,5} - 2\zeta_2 f_{3,3,3,3} + \frac{5611}{132}\zeta_8 f_{3,3}\Big].$$

The values of the NMHV amplitudes $E^{(L)}(1,1,1)$ for $L = 1$ to 6 in the f-basis are

$$
\begin{aligned}
E^{(1)}(1,1,1) &= -2\,\zeta_2\,,\\
E^{(2)}(1,1,1) &= 26\,\zeta_4\,,\\
E^{(3)}(1,1,1) &= -\frac{940}{3}\,\zeta_6\,,\\
E^{(4)}(1,1,1) &= \frac{36271}{9}\,\zeta_8 - 24\left[5f_{3,5} - 2\zeta_2 f_{3,3}\right],\\
E^{(5)}(1,1,1) &= -\frac{1666501}{30}\,\zeta_{10} + 528\left[7f_{3,7} - \zeta_2 f_{3,5} - 3\zeta_4 f_{3,3}\right] + 384\left[5f_{5,5} - 2\zeta_2 f_{5,3}\right],\\
E^{(6)}(1,1,1) &= \frac{5066300219}{6219}\zeta_{12} - 4664\left[7f_{3,9} - 6\zeta_4 f_{3,5}\right] - 11384\left[5f_{3,9} - 3\zeta_6 f_{3,3}\right]\\
&\quad + 5664\left[\zeta_2 f_{3,7} - \zeta_6 f_{3,3}\right] - 8928\left[7f_{5,7} - \zeta_2 f_{5,5} - 3\zeta_4 f_{5,3}\right]\\
&\quad - 6528\left[5f_{7,5} - 2\zeta_2 f_{7,3}\right].
\end{aligned}
$$

Tiny caveat

- To squeeze amplitudes into a space $\mathcal{H}^{\text{hex}}$ that obeys a co-action principle, we need to adjust their normalization slightly:

$$\mathcal{E} \to \frac{\mathcal{E}}{\rho}, \qquad E \to \frac{E}{\rho}$$

$$\begin{aligned}
\rho(g^2) \,=\, 1 + 8(\zeta_3)^2\, g^6 - 160\zeta_3\zeta_5\, g^8 + \Big[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\Big]\, g^{10} \\
- \Big[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\Big]\, g^{12} \\
+ \Big[221760\zeta_3\zeta_{11} + 247296\zeta_5\zeta_9 + 126240(\zeta_7)^2 - 3360\zeta_4\zeta_3\zeta_7 - 1824\zeta_4(\zeta_5)^2 \\
- 5440\zeta_6\zeta_3\zeta_5 - 4480\zeta_8(\zeta_3)^2\Big]\, g^{14} \;+\; \mathcal{O}(g^{16}).
\end{aligned}$$

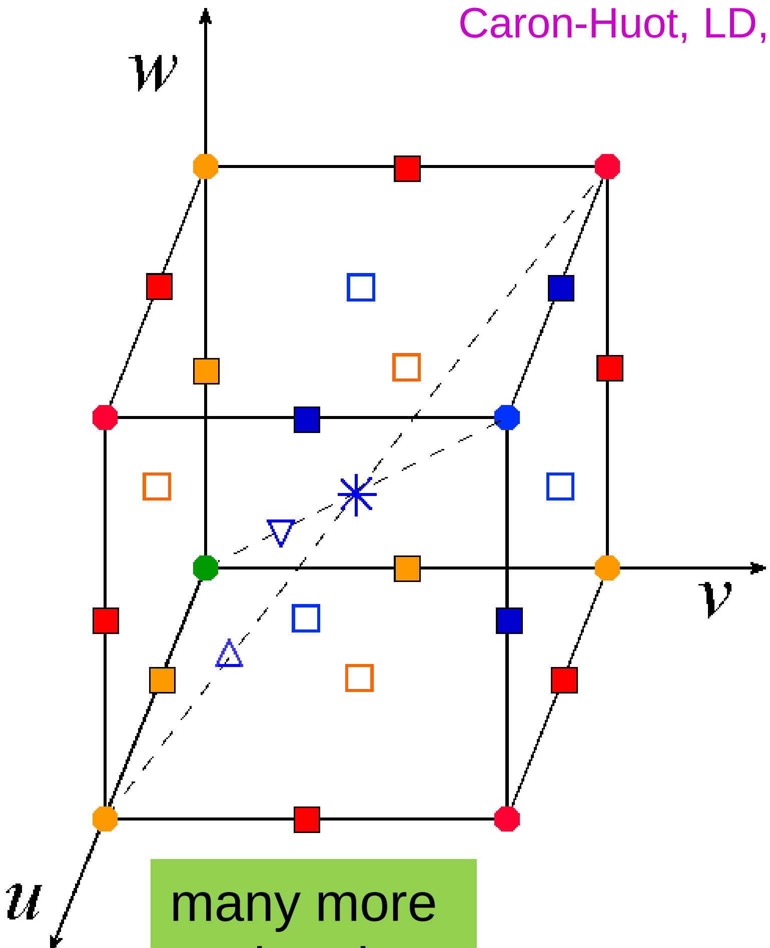
- First we found $\rho(g^2)$ empirically, order by order.
- Now we have an all-orders formula for an improved version of it

in terms of the BES kernel Basso, LD, Papathanasiou, 2001.05460

controlling the cusp anomalous dimension Beisert, Eden, Staudacher (2006)

6-gluon amplitude
→ many “cyclotomic” polylogs at unity

Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1905.nnnnn;
Ablinger, Blumlein, Schneider, 1105.6063, 1310.5645;
O. Schnetz, **HyperlogProcedures**



● MZVs
■, □ Alternating sums
✳ 4th roots of unity
▽, △ 6th roots of unity

finite
1 variable singular
2 variables singular
3 variables singular

Co-action principle applies to entire function space at every point where we’ve checked it!!

many more such points!

e.g.

$$u = v = w, \qquad y_u = y_v = y_w = y,$$
$$u = \frac{y}{(1+y)^2} \qquad 1 - u = \frac{1+y+y^2}{(1+y)^2}$$

Saturation

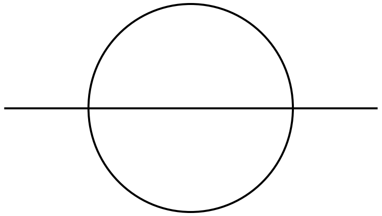
- Take iterated $\{n\text{-}1,1\}$ coproducts of these amplitudes → generate more and more lower weight functions until space is "saturated" and number declines again

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$L = 1$	1	3	4												
$L = 2$	1	3	6	10	6										
$L = 3$	1	3	6	13	23	15	6								
$L = 4$	1	3	6	13	27	50	50	24	6						
$L = 5$	1	3	6	13	27	54	97	117	70	24	6				
$L = 6$	1	3	6	13	27	54	102	188	255	179	78	24	6		
$L = 7$	1	3	6	13	27	54	102	190	337	490	409	209	79	24	6
$L \leq 7$	1	3	6	13	27	54	102	190	337	490	416	219	82	24	6

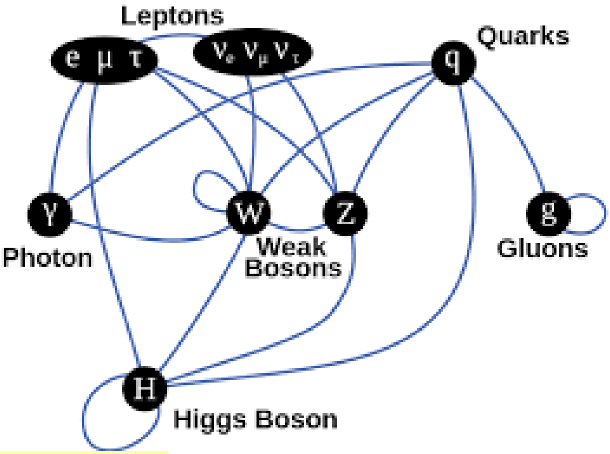
- Bottom up construction of space:

1 3 6 13 27 54 105 200 372 679 1214 2136 …

ϕ^4 theory



Theory of Higgs boson, neglecting all other Standard Model couplings.



Pure $O(N)$ symmetric ϕ^4 theory in $D = 4 - 2\varepsilon$ experimentally relevant for ε expansion approach to critical exponents in $D = 3$

Wilson, Fisher (1972); Guillou, Zinn-Justin; Kleinert, Vasil'ev,…

High order computations required since $\varepsilon = 1/2$

- ε expansion recently completed to 6 loops

→ 3-4 digits accuracy for critical exponents after Borel resummation

Kompaniets, Panzer, 1705.06483

- Many primitive divergences known to much higher orders.

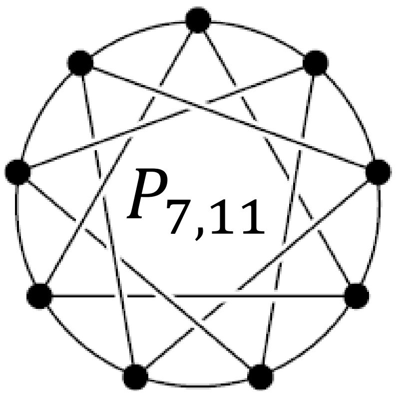
Co-action principle in ϕ^4 theory

- Earlier: Hopf algebra associated with nested structure of renormalization; knots and Feynman diagrams
  Broadhurst, Kreimer, hep-th/9504352, hep-th/9810087
- Co-action principle first formulated for ϕ^4 theory
- Much data now for primitive graphs, those with no subdivergences

  Schnetz, 1302.6445; Panzer, Schnetz, 1603.04289

Panzer, Schnetz, 1603.04289

- "Period" = UV divergence of ϕ^4 graph containing no subdivergences



- Here, co-action principle works "graph by graph", i.e. result of clipping f_i on left is a the period for a subgraph of original graph

Proof: Brown, 1512.06409

In the following table we demonstrate that the known ϕ^4 periods up to eight loops obey the coaction conjecture. For this we express the infinitesimal coaction in terms of ϕ^4 periods.

period	$\sum_m f_m^N \delta_m(P_\bullet)$
P_1	0
P_3	$6 f_3 P_1$
P_4	$20 f_5 P_1$
P_5	$\frac{441}{8} f_7 P_1$
$P_{6,1}$	$168 f_9 P_1$
$P_{6,2}$	$\frac{2}{3} f_3 P_3^2 + \frac{1\,063}{9} f_9 P_1$
$P_{6,3}$	$\frac{63}{5} f_3 P_4 - 30 f_5 P_3$
$P_{6,4}$	$-\frac{648}{5} f_3 P_4 + 720 f_5 P_3$
$P_{7,1}$	$\frac{33\,759}{64} f_{11} P_1$
$P_{7,2}$	$\frac{7}{12} f_3 P_3 P_4 - \frac{5}{18} f_5 P_3^2 - \frac{195\,379}{192} f_{11} P_1$
$P_{7,3}$	$\frac{1}{3} f_3 P_3 P_4 - \frac{31}{9} f_5 P_3^2 - \frac{960\,211}{240} f_{11} P_1$
$P_{7,4}, P_{7,7}$	$\frac{160}{21} f_3 P_5 - 20 f_5 P_4 + 70 f_7 P_3$
$P_{7,5}, P_{7,10}$	$-\frac{24}{7} f_3 P_5 + 45 f_5 P_4 - \frac{63}{2} f_7 P_3$
$P_{7,6}$	$\frac{7}{12} f_3 P_3 P_4 + \frac{145}{18} f_5 P_3^2 + \frac{502\,247}{64} f_{11} P_1$
$P_{7,8}$	$f_3(7 P_{6,3} - \frac{161}{30} P_3 P_4) + \frac{527}{9} f_5 P_3^2 + \frac{2\,756\,439}{20} f_{11} P_1$
$P_{7,9}$	$f_3(\frac{7}{2} P_{6,3} - \frac{133}{80} P_3 P_4) - \frac{217}{24} f_5 P_3^2 + \frac{4\,136\,619}{160} f_{11} P_1$
$P_{7,11}$	$f_2^6(-\frac{2\,755}{864} P_{6,1} + \frac{35}{27} P_3^3) + \frac{14}{9} f_4^6 P_5 + \frac{1\,017}{22} f_6^6 P_4 - \frac{36\,918}{43} f_8^6 P_3$
$P_{8,1}$	$1716 f_{13} P_1$
$P_{8,2}$	$f_3(\frac{145}{147} P_3 P_5 - \frac{27}{80} P_4^2) + \frac{29}{40} f_5 P_3 P_4 + \frac{47}{16} f_7 P_3^2 + \frac{94\,871\,691}{22\,400} f_{13} P_1$
$P_{8,3}$	$f_3(2 P_4^2 - \frac{320}{189} P_3 P_5) - 13466 f_{13} P_1$
$P_{8,4}$	$f_3(\frac{27}{80} P_4^2 + \frac{1}{147} P_3 P_5) + \frac{11}{40} f_5 P_3 P_4 - \frac{97}{16} f_7 P_3^2 - \frac{76\,207\,221}{22\,400} f_{13} P_1$
$P_{8,5}$	$\frac{789}{112} f_3 P_{6,1} - \frac{2\,930}{147} f_5 P_5 + \frac{3\,549}{40} f_7 P_4 - 180 f_9 P_3$
$P_{8,6}, P_{8,9}$	$\frac{488}{441} f_3 P_3 P_5 - \frac{29}{2} f_7 P_3^2 - \frac{1\,717\,423}{336} f_{13} P_1$
$P_{8,7}, P_{8,8}$	$-\frac{81}{10} f_5 P_3 P_4 + \frac{75}{4} f_7 P_3^2 - \frac{9\,819\,147}{2\,800} f_{13} P_1$

Summary

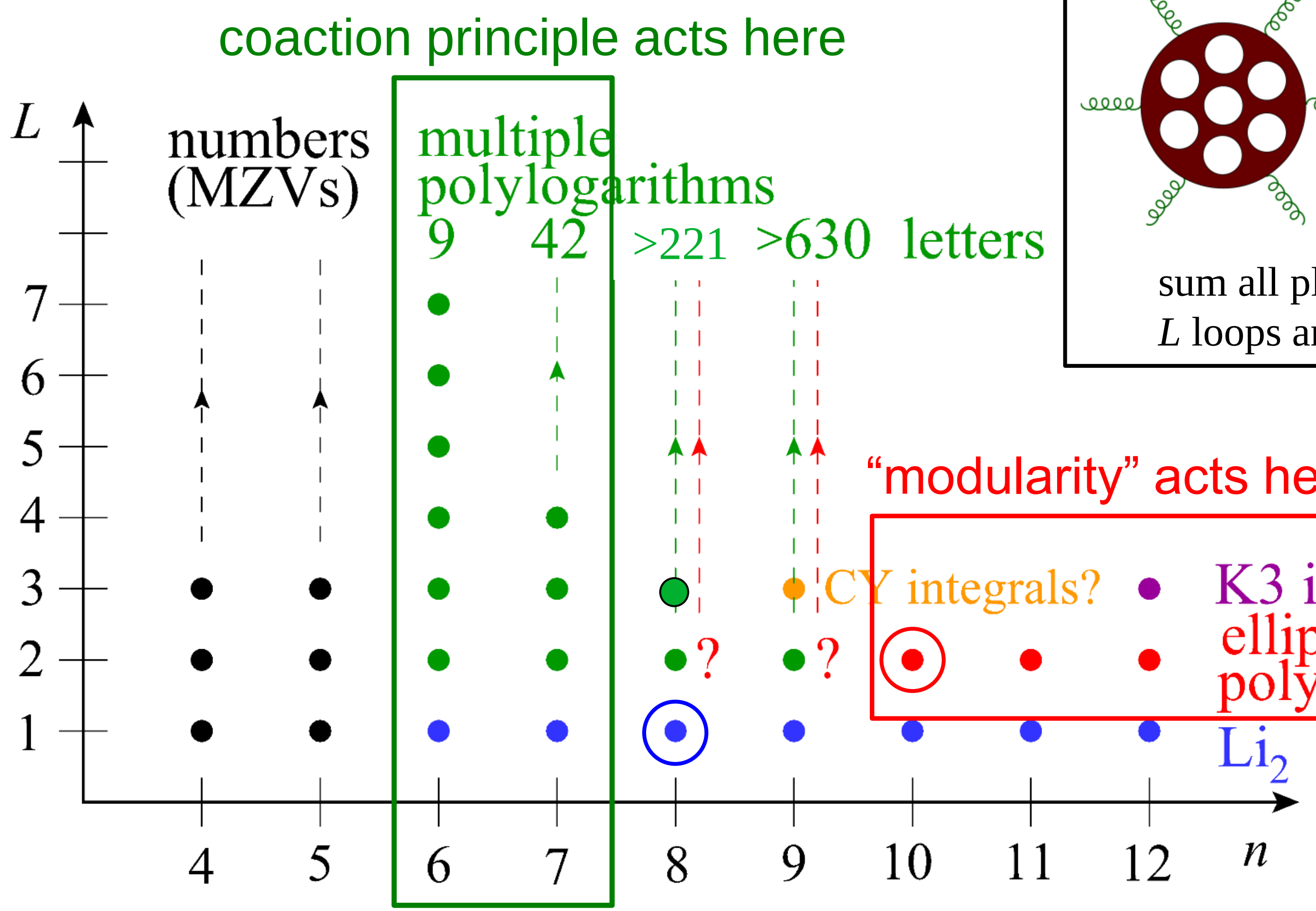
- Euler was on to something, 270 years ago!
- Many important physical quantities expressed in terms of the (conjecturally) transcendental MZVs he introduced, and related generalizations.
- Properties of numbers unveiled by embedding them into (polylogarithmic) functions with an associated Hopf co-algebra
- Whenever there is a lot of theoretical data – g_e-2, planar N=4 SYM amplitudes, ϕ^4 theory – the relevant numbers appear to obey a co-action principle.

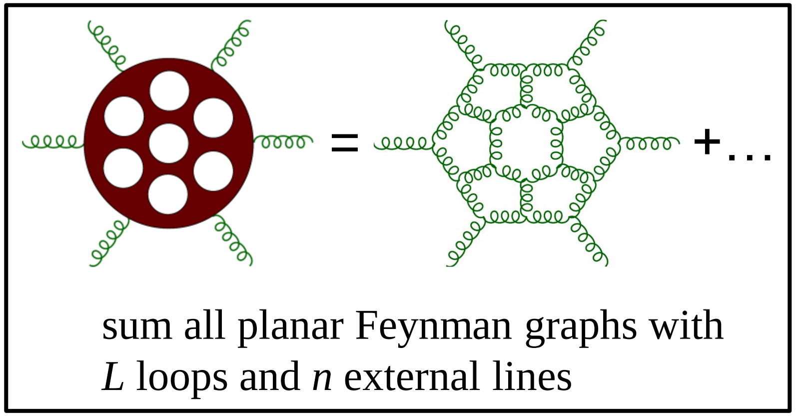
Outlook

- In many cases, polylogarithms and MZVs do not suffice for multi-loop Feynman integrals
  - need elliptic polylogarithms or "worse".
- How exactly the co-action works there is still in its infancy
- To how many arenas of QFT can these ideas be applied?
- Does any general principle lurk behind what is there (including the rational numbers??) as well as what is not there?

Extra Slides

One context:
Loop amplitudes in planar N=4 SYM depend on $3(n\text{-}5)$ variables





Co-action for QCD scattering amplitudes?

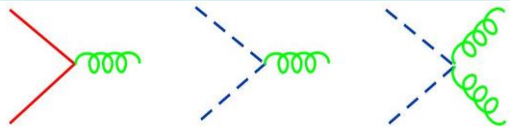
- Same Galois conjugates for g_e-2 appear in quark (chromo) magnetic moments through 3 loops, also q^2 dependence of form factors Bonciani, Mastrolia, Remiddi, hep-ph/0307295; Lee, Smirnov, Smirnov, Steinhauser, 1801.08151, 1804.07310; …
- Also evidence for interesting number theory in QCD β function, e.g. no $\pi' s$ until 5 loops, when π^4 appears; predictions of π dependence at 6,7 loops Baikov, Chetyrkin, Kühn, 1606.08659; Baikov, Chetyrkin, 1804.10088, 1808.00237
- Unfortunately, know very few full QCD amplitudes beyond two loops, where co-action principle becomes more predictive.
- Can say a lot more for QCD's maximally supersymmetric cousin, N=4 supersymmetric Yang Mills theory (N=4 SYM), especially in (planar) limit of a large number of colors where it has many secret symmetries.

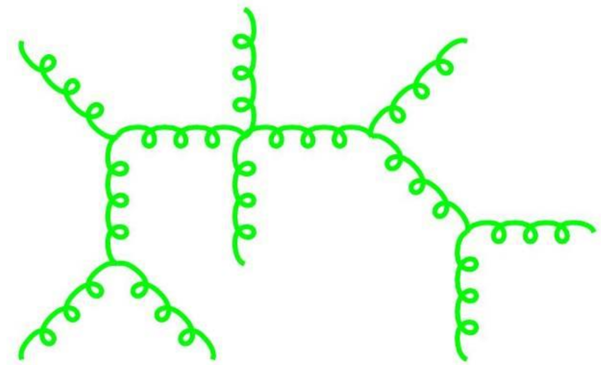
How are QCD and N=4 SYM related?

At tree level they are essentially identical

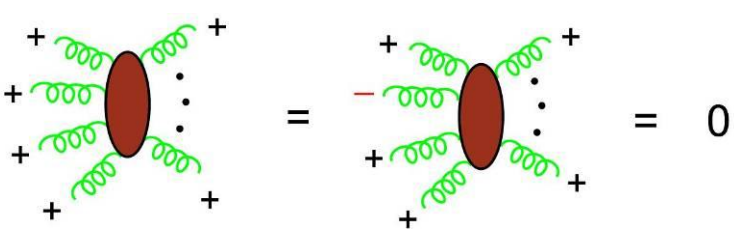
Consider a tree amplitude for *n* gluons.

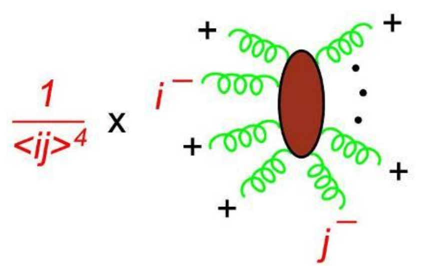
Fermions and scalars cannot appear because they are produced in pairs





Hence the amplitude is the **same** in QCD and N=4 SYM.
So the QCD tree amplitude "secretly" obeys
all identities of N=4 supersymmetry:





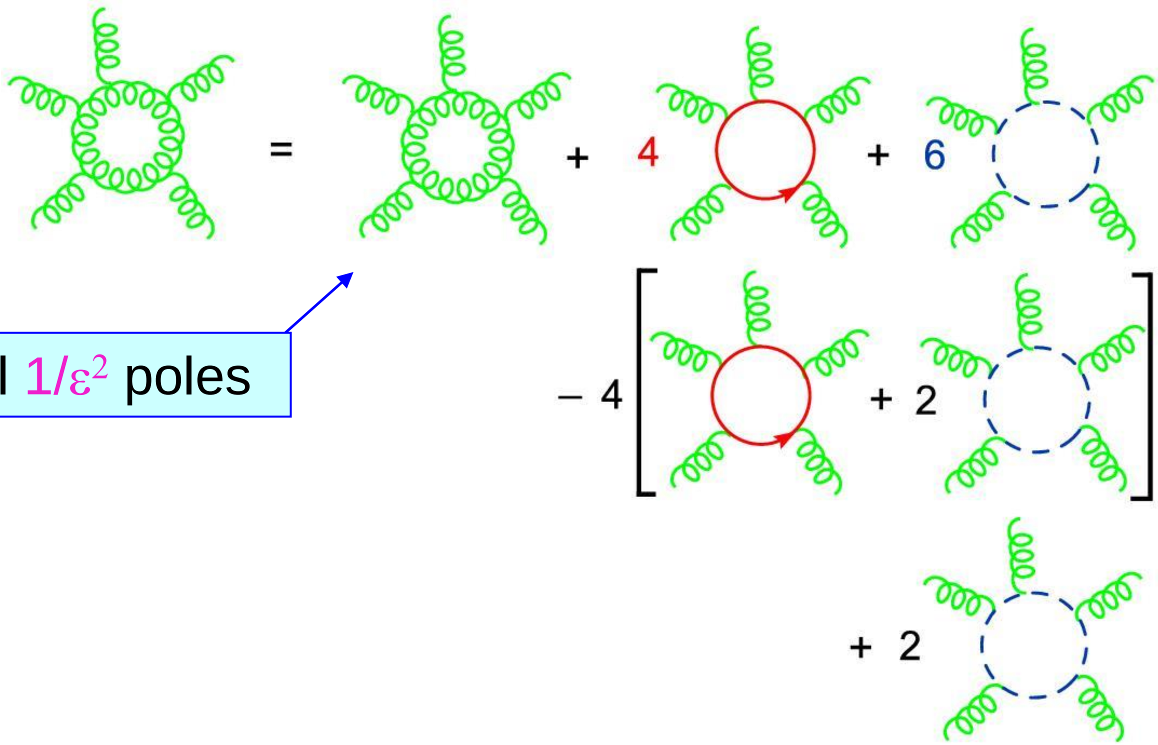
$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

independent of *i,j*

At loop level, QCD and N=4 SYM differ

However, it is profitable to rearrange the QCD computation to exploit supersymmetry

gluon loop



N=4 SYM

N=1 multiplet

scalar loop --- no SUSY, but also no spin tangles

contains all $1/\varepsilon^2$ poles

Strong coupling and soap bubbles

Alday, Maldacena, 0705.0303

- Use AdS/CFT to compute scattering amplitude
- High energy scattering in string theory semi-classical: two-dimensional string world-sheet stretches a long distance, classical solution minimizes area

Gross, Mende (1987,1988)

Classical action imaginary
→ exponentially suppressed tunnelling configuration

$$A_n \sim \exp[-\sqrt{\lambda} S_{\text{cl}}^{\text{E}}]$$