Scattering Amplitudes

George Goulas

November 3, 2021

Spinor-Helicity Formalism

Consider the Dirac equation for a 4-component spinor:

$$(i\partial\!\!\!/ - m)\psi(x) = 0 \tag{1}$$

The general solution is a superposition of plane waves:

$$\psi(x) \sim u(p)e^{ipx} + v(p)e^{-ipx} \tag{2}$$

Assuming that the spinors u,v satisfy:

$$\begin{aligned} (p\!\!\!/ + m)u(p) &= 0 \\ (p\!\!\!/ - m)v(p) &= 0 \end{aligned} \tag{3}$$

We already know that these describe fermions and anti-fermions respectively. In fact $\bar{v}, u$ are associated with incoming (anti)fermions and $v, \bar{u}$ are associated with outgoing (anti)fermions.

Spinor-Helicity Formalism

Let's focus on the spinors describing outgoing particles, $v, \bar{u}$ and go to the massless/high energy limit. The Dirac equation becomes:

$$\begin{aligned} \bar{u}_\pm(p)\not{p} &= 0 \\ \not{p}v_\pm(p) &= 0 \end{aligned} \tag{4}$$

In the above, the subscript denotes helicity $h = \pm\frac{1}{2}$ depending on our choice. we write the two independent solutions to the Dirac equation as:

$$\begin{aligned} v_+(p) &= \begin{bmatrix} |p]_\alpha \\ 0 \end{bmatrix} \\ v_-(p) &= \begin{bmatrix} 0 \\ |p\rangle^{\dot{\alpha}} \end{bmatrix} \\ \bar{u}_-(p) &= \begin{bmatrix} 0 & \langle p|_{\dot{\alpha}} \end{bmatrix} \\ \bar{u}_+(p) &= \begin{bmatrix} [p|^\alpha & 0 \end{bmatrix} \end{aligned} \tag{5}$$

Spinor-Helicity Formalism

We thus have introduced two component spinors, the angle spinors $|p\rangle^{\dot{\alpha}}$ and the square spinors $[p|^{\alpha}$. The indices are raised and lowered with the Levi-Civita symbol: $[p|^{\alpha} = \epsilon^{\alpha\beta}|p]_{\beta}$.
We also introduce the angle and square spinor brakets as:

$$\begin{aligned} \langle p|q\rangle &= \langle p|_{\dot{\alpha}} \, |q\rangle^{\dot{\alpha}} \\ [pq] &= [p|^{\alpha}|q]_{\alpha} \\ \langle p|q\rangle \, [pq] &= 2p \cdot q = (p+q)^2 \end{aligned} \tag{6}$$

All other brakets vanish, eg$\langle q| \, |p]$. For real momenta these spinors are not independent, but instead satisfy:

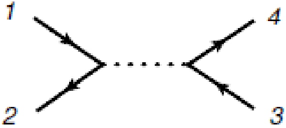
$$\begin{aligned} [p|^{\alpha} &= (|p\rangle^{\dot{\alpha}})^{\star} \\ \langle p|_{\dot{\alpha}} &= (|p]_{\alpha})^{\star} \\ [pq]^{\star} &= \langle q|p\rangle \end{aligned} \tag{7}$$

Spinor-Helicity Formalism

Let's see a few examples of scattering amplitudes using the spinor-helicity formalism, starting from the Yukawa theory. The Lagrangian is:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi + g\phi\bar{\psi}\psi \tag{8}$$

which involves one vertex associated with ig. We consider the amplitude $A_4(\bar{f}^{h1} f^{h_2} \bar{f}^{h3} f^{h_4})$. This is a helicity amplitude, since all involved particles have definite helicity. The s-channel diagram is:



$$= ig\,\bar{u}_4 v_3 \frac{-i}{(p_1+p_2)^2} ig\,\bar{u}_2 v_1$$

We can already see that this vanishes unless particles 1(3) and 2(4) have the same helicity. In the opposite case, we would be dealing with a vanishing braket.

Spinor-Helicity Formalism

Assume that particles 1,2 have negative helicity and 3,4 have positive. Then the above diagram is the only contribution to this process and after plugging in the appropriate spinors we get:

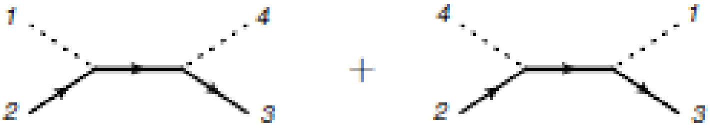
$$iA_4(\bar{f}^- f^- \bar{f}^+ f^+) = ig^2 [43] \frac{1}{2p_1 \cdot p_2} \langle 2|1\rangle = ig^2[43]\frac{1}{\langle 2|1\rangle\,[12]}\langle 2|1\rangle = ig^2\frac{[34]}{[12]} \tag{9}$$

Which is a nice ratio of two spinor brackets. Using momentum conservation $p_1 + p_2 = p_3 + p_4$ we can get another form, namely:

$$iA_4(\bar{f}^- f^- \bar{f}^+ f^+) = ig^2 \frac{\langle 1|2\rangle}{\langle 3|4\rangle} \tag{10}$$

Spinor-Helicity Formalism

Another example is the scattering amplitude between two scalars and two fermions $iA(\phi \bar{f}^{h_1} f^{h2} \phi)$. The two relevant diagrams are:



$$= (ig)^2 \bar{u}_3 \frac{-i(\not{p}_1 + \not{p}_2)}{(p_1 + p_2)^2} v_2 + 1 \leftrightarrow 4 \tag{11}$$

The nominator has a product involving the γ^μ matrices, which forces the incoming and outgoing spinors to have opposite helicity:

Spinor-Helicity Formalism

$$\bar{u}_{-}(p_3)\gamma^{\mu}v_{+}(p_2) = \begin{bmatrix} 0 & \langle 3|_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} 0 & \sigma^{\mu}_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{bmatrix} \begin{bmatrix} |2]_{\beta} \\ 0 \end{bmatrix} \tag{12}$$

It is not difficult to see from this that spinors with the same helicity make this vanish, i.e. $\bar{u}_{-}(p_3)\gamma^{\mu}v_{-}(p_2) = 0$. One then can verify that:

$$A(\phi\bar{f}^{+}f^{-}\phi) = -g^2\left(\frac{\langle 1|3\rangle}{\langle 1|2\rangle} + \frac{\langle 3|4\rangle}{\langle 2|4\rangle}\right) \tag{13}$$

Spinor-Helicity Formalism

Finally let's see an application of the spinor helicity formalism to QED, where there are massless vectors involved. The Lagrangian is the familiar:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} - ieA_{\mu})\psi \tag{14}$$

The vertex is the familiar $-ie\gamma^{\mu}$. The rule for external photons is to represent them with a polarization vector $\epsilon^{\mu}_{\pm}$. We can represent these polarization vectors as:

$$\begin{aligned}
\epsilon^{\mu}_{-}(p;q) &= -\frac{\langle p|\,\gamma^{\mu}|q]}{\sqrt{2}[qp]} \\
\epsilon^{\mu}_{+}(p;q) &= -\frac{\langle q|\,\gamma^{\mu}|p]}{\sqrt{2}\,\langle q|p\rangle}
\end{aligned} \tag{15}$$

The q above is an arbitrary reference spinor and as such will drop out of any final result. It reflects gauge invariance, because one can shift $\epsilon^{\mu}_{\pm}(p) \to \epsilon^{\mu}_{\pm}(p) + Cp^{\mu}$ and not change the s.a. since $p_{\mu}A^{\mu} = 0$.

Spinor-Helicity Formalism

Consider for instance $A_3(f^{h_1}\bar{f}^{h_2}f^{h_3})$, involving a photon, an electron and a positron. Choose for instance $h_1 = -\frac{1}{2}, h_2 = \frac{1}{2}, h_3 = -1$. Then we have:

$$iA_3(f^-\bar{f}^+\gamma^-) = \bar{u}_-(p_1)ie\gamma_\mu v_+(p_2)\epsilon_-^\mu(p_3;q) = -ie\,\langle 1|\,\gamma_\mu|2]\frac{\langle 3|\,\gamma^\mu|q]}{\sqrt{2}[3q]}$$

A bit of algebra involving identities of the angle and square spinors leads to:

$$A_3(f^-\bar{f}^+\gamma^-) = \sqrt{2}e\frac{\langle 1|3\rangle^2}{\langle 1|3\rangle} \tag{16}$$

The fact that the final result depends only on angle brackets is no coincidence but a consequence of 3-particle special kinematics. Any 3-point on shell amplitude with massless particles depends only on either square brackets or angle brackets of the external momenta.

Colour-Ordered Amplitudes

We now proceed to consider gluon scattering. The Lagrangian describing gluons is the usual:

$$\mathcal{L} = -\frac{1}{4} F^{\alpha}_{\mu\nu} F^{\alpha\mu\nu} \tag{17}$$

We consider a general SU(N) group, whose algebra is:

$$\left[T^a, T^b\right] = if^{abc} T^c \tag{18}$$

In order to extract the Feynman rules and the propagator we choose the Gervais-Neveu gauge, in which the Lagrangian takes the form:

$$\mathcal{L} = Tr(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - i\sqrt{2} g \partial^\mu A^\nu A_\nu A_\mu + \frac{g^2}{4} A^\mu A^\nu A_\nu A_\mu) \tag{19}$$

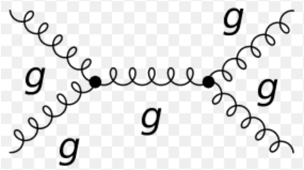
The gluon propagator in this case is simply: $\Delta^{\alpha\beta}_{F\mu\nu}(p) = \delta^{\alpha\beta} \frac{\eta_{\mu\nu}}{p^2}$

Colour-Ordered Amplitudes

Consider now a tree-level scattering amplitude. In general, this involves products of the structure constants and generators of the gauge-group but can always be arranged in the form:

$$A_n^{tree} = g^{n-2} \sum_{perms \sigma} A_n[1\sigma(2...n)]\, Tr(T^{a_1} T^{\sigma(a_2} ... T^{a_n)}) \tag{20}$$

The partial amplitudes involved in the above sum are called colour ordered amplitudes and are gauge invariant. As an example to this, consider the s channel tree-level 4-gluon amplitude.



$$\sim f^{a_1 a_2 b} f^{b a_3 a_4}$$

Colour-Ordered Amplitudes

We can use the completeness relation $if^{abc} = Tr(T^aT^bT^c) - Tr(T^bT^aT^c)$ as well as the completeness relation $(T^a)_i^j(T^a)_k^l = \delta_i^l\delta_k^j - \frac{1}{N}\delta_i^j\delta_k^l$ to write:

$$
\begin{aligned}
&f^{a_1a_2b}f^{ba_3a_4} \sim \\
&Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - Tr(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) - Tr(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) \\
&+ Tr(T^{a_1}T^{a_4}T^{a_3}T^{a_2})
\end{aligned}
\tag{21}
$$

Notice the cyclic permutations of indices 2,3,4. This means that the total amplitude can be written as:

$$
A_4^{tree} = g^2(A_4[1234]Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + perms(2,3,4)) \tag{22}
$$

which is of the form of eq. 20. So we only need to calculate one amplitude in order to get our answer. These diagrams are computed without any crossing lines and with the indices fixed in the order written.

Colour-Ordered Amplitudes

The Feynman rules for colour ordered amplitudes are:

$$\begin{aligned} V^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) &= -\sqrt{2}(\eta^{\mu_1\mu_2} p_1^{\mu_3} + \eta^{\mu_2\mu_3} p_2^{\mu_1} + \eta^{\mu_3\mu_1} p_3^{\mu_2}) \\ V^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4) &= \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} \end{aligned} \tag{23}$$

The polarization vectors are the same as in QED. One can see the power of this technology when considering for instance the 3-gluon scattering amplitude:

$$A_3[123] = -\sqrt{2}[(\epsilon_1\epsilon_2)(\epsilon_3 p_1) + (\epsilon_2\epsilon_3)(\epsilon_1 p_2) + (\epsilon_3\epsilon_1)(\epsilon_2 p_3)] \tag{24}$$

By using 3-particle special kinematics and properties of the spinor-helicity formalism, one can find that the non-vanishing colour-ordered helicity amplitudes are:

$$A_3(1^-2^-3^+) = \frac{\langle 1|2\rangle^3}{\langle 2|3\rangle\,\langle 3|1\rangle} \quad A_3(1^+2^+3^-) = \frac{[12]^3}{[23][31]} \tag{25}$$

Colour-Ordered Amplitudes

Another powerful result can be obtained when one considers n gluons scattering, with i,j having helicity -1 and the other n-2 have helicity +1. This is the so-called Parke-Taylor n-gluon tree amplitude and is given by:

$$A_n[1^+ \ldots i^- \ldots j^- \ldots n^+] = \frac{\langle i|j\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle \ldots \langle n|1\rangle} \tag{26}$$

Thus the 4-gluon tree level amplitude for example would be:

$$A_4[1^- 2^- 3^+ 4^+] = \frac{\langle 1|2\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle \langle 3|4\rangle \langle 4|1\rangle} \tag{27}$$

MHV Amplitudes

One can show by using dimensional analysis and a proper choice of polarization vectors, that $A_n(1^+2^+ ... n^+) = 0$. The same holds true for $A_n(1^-2^+ ... n^+) = 0$. The first non vanishing amplitude is in fact of the form of eq 26 $A_n(1^-2^-3^+ ... n^+)$. This is known as a maximally helicity violating-MHV amplitude. The MHV gluon amplitudes are the simplest amplitudes in Yang-Mills theory.
Flipping one more helicity we obtain $A_n(1^-2^-3^- ... n^+)$ which is a NMHV amplitude (next to MHV). The notion generalizes to N^kMHV amplitudes, which have k+2 negative helicities and n-k-2 positive ones. On the contrary, if k=n we have an anti-MHV amplitude which has n-2 negative helicities and exactly two positive ones. This can easily be obtained from the MHV amplitude with all helicities flipped by simply exchanging angle brackets with square brackets.
Another important result is that N^kMHV amplitudes can be written as a sum of all tree-level diagrams with precisely K+1 MHV vertices-this is known as the MHV vertex expansion or CSW expansion. It is based on recursion relations that exist among scattering amplitudes.

Recursion Relations

Another approach to studying scattering amplitudes is to use tools from complex analysis. One starts by considering shifted momenta on an amplitude:

$$\hat{p}_i^\mu = p_i^\mu + z r_i^\mu \tag{28}$$

where the vectors r_i^μ are chosen with desirable properties, such as $\sum_i r_i^\mu = 0$ which guarantees momentum conservation for the shifted momenta. We want to consider the shifted amplitude $\hat{A}_n(z)$ as a function of z.
A generic propagator in a tree-level graph involves $\sim 1/P_I^2$ where $P_I^\mu = \sum_{i \in I} p_i^\mu$. By choosing r_i^μ to be light-like, one can show that

$$\hat{P_I}^2 = -\frac{P_I^2}{z_I}(z - z_i), \quad z_I = -\frac{P_I^2}{2P_I \cdot R_I} \tag{29}$$

Thus the poles of $\hat{A}_n(z)$ are simple poles and for generic momenta they are all located away from the origin.

Recursion Relations

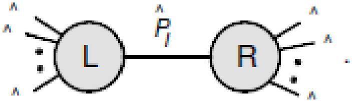
Consider now $\frac{\hat{A}_n(z)}{z}$. This has simple poles at $z = z_I$ and one pole at $z = 0$. The residue at the origin is the just the original amplitude A_n. Thus, from Cauchy's theorem we get:

$$A_n = -\sum_{z_I} Res_{z=z_I}(\frac{\hat{A}_n(z)}{z}) + B_n \tag{30}$$

The B_n is a contribution coming from the pole at infinity. Usually one assumes or proves this to be 0. The reason for all the above is that at each of the z_I poles, the propagator goes on shell and the amplitude factorizes into two subamplitudes:

$$Res_{z=z_I}\frac{\hat{A}_n(z)}{z} = -\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) \tag{31}$$

Recursion Relations



Factorization of the amplitude at the pole z_I

The rule for the internal line in the above graph is to just write the propagator of the unshifted momenta $1/P_I^2$. Each of the two subamplitudes involves less than n particles, and is thus easier to compute. Provided that $B_n \to 0$, the shift is called good and then n-point amplitude is completely determined in terms of lower point on-shell amplitudes.

$$A_n = \sum_{diagrams I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) \tag{32}$$

which is the most general form of a recursion relation.

Recursion Relations

The most famous of the recursion relations is the so-called BCFW relation. In that, one shifts just two of the involved momenta, say i and j. In terms of angle and square spinors, this is expressed as:

$$\begin{aligned} [\hat{i}] &= [i] + z[j] \quad & \left|\hat{j}\right\rangle &= |j\rangle - z\,|i\rangle \\ [\hat{j}] &= [j] & \left|\hat{i}\right\rangle &= |i\rangle \end{aligned} \tag{33}$$

One can use this for instance to construct an inductive proof of the Parke-Taylor n-gluon tree amplitude formula we presented earlier (26).

Recursion Relations

Recursion relations can be applied to various field theories in order to obtain the tree-level scattering amplitudes. Some examples include:

- Yang-Mills theories: we already got a sense of it by studying gluon amplitudes. An amazing feature is that the information needed for the amplitudes is fully captured by the cubic term $AA\partial A$, without needing to involve the quartic vertex A^4.
- $\mathcal{N} = 4$ SYM theory. When SUSY is incorporated in the BCFW recursion relations all tree amplitudes of this theory can be determined by the 3-point gluon vertex alone.
- Gravity: Because of the validity of BCFW recursion relations, the entire on-shell S-matrix for gravity is fixed by the 3-graviton vertex. The infinite terms appearing in the EH action $S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R$ exists just to ensure diffeomorphism invariance of the off-shell Lagrangian.

Supersymmetry

Supersymmetry transformations map bosons to fermions and vice versa. To get a visual of this consider a Weyl fermion and a complex scalar field:

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \partial_\mu \bar{\phi} \partial^\mu \phi \tag{34}$$

Up to a total derivative, the transformations:

$$\delta_\epsilon \phi = \epsilon \psi, \quad \delta_\epsilon \psi_\alpha = -i\sigma^\mu_{\alpha\dot{\beta}} \epsilon^{\dagger\dot{\beta}} \partial_\mu \phi \tag{35}$$

leave the Lagrangian unchanged. Consider furthermore an interaction Lagrangian of the form:

$$\mathcal{L}_I = \frac{1}{2} g \phi \psi \psi + \frac{1}{2} g^\star \bar{\phi} \psi^\dagger \psi^\dagger - \frac{1}{4} |g|^2 |\phi|^4 \tag{36}$$

Scattering amplitudes with external states related by SUSY are related to each other through linear relationships called supersymmetric Ward identities. Let's see an example:

Supersymmetry

Assuming a supersymmetric vacuum $Q\left|0\right\rangle = Q^\dagger\left|0\right\rangle = 0$, the following v.e.v. vanishes:

$$\langle 0|\left[Q^\dagger, a_-(p_1)b_-(p_2)a_+(p_3)a_+(p_4)\right]|0\rangle = 0 \tag{37}$$

The a operators are associated with the scalar field in our Lagrangian and b with the fermion. Expanding the r.h.s. and using some SUSY algebra we obtain:

$$0 = \left|2\right\rangle A_4(\phi\phi\bar{\phi}\bar{\phi}) - \left|3\right\rangle A_4(\phi f^- f^+ \bar{\phi}) - \left|4\right\rangle A_4(\phi f^- \bar{\phi} f^+) \tag{38}$$

Then by dotting the above with proper bra spinors (and recalling that momenta are null) we can obtain various relations such as:

$$A_4(\phi f^- \bar{\phi} f^+) = -\frac{\langle 2|3\rangle}{\langle 2|4\rangle} A_4(\phi f^- f^+ \bar{\phi}) \tag{39}$$

Supersymmetry

Most of the technology introduced earlier can be properly extended to supersymmetric theories. We can define for instance chiral superfields and superamplitudes and process these using supersymmetric versions of the BCFW shift or the MHV vertex expansion with many of the result holding true in this case as well.
For instance, we have already seen that at tree level the amplitudes $A_n(1^+2^+...n^+) = A_n(1^-2^+...n^+) = 0$. If we include supersymmetry then the corresponding amplitudes in SYM vanish to all orders in perturbation theory:

$$A_n^{L-loop}(g^+g^+...g^+) = A_n^{L-loop}(g^-g^+...g^+) = 0 \tag{40}$$

The end

Thank you for your attention.