ITS Symposium
on the Categorical
Semantics of Entropy
(CUNY) 5/13/2022

Higher Entropy

What is "Higher Entropy?"

How much ignorance do I have? $\rightsquigarrow$ What is it that I don't know?

(relative) Entropy $\rightsquigarrow$ "Linear Space" of random Variables

measuring these resolves ignorance

E.g. $\hat{\mu} : \Omega \longrightarrow \mathbb{R}_{\geq 0}$

$$S(\hat{\mu}) = \sum_{\omega \in \Omega} \hat{\mu}_\omega \log \hat{\mu}_\omega \rightsquigarrow L^q(\Omega_{\mu \neq 0})$$

$q \in \mathbb{H}_{\mathrm{Re}\, q \geq 1}$, $\mathbb{C}$

• $\|f\|_q = \left| \sum_\omega (f^*_\omega f_\omega)^{q/2} \right|^{1/q}$

Homotopical Perspective: Study multipartite Systems
e.g. $\mu_{AB} : \Omega_A \times \Omega_B \longrightarrow \mathbb{R}_{\geq 0}$

How much info. is shared? $\rightsquigarrow$ What info. is shared?

Mutual Information:

$$I(\mu_{AB}, \{A, B\}) = S_A + S_B - S_{AB} \rightsquigarrow \text{"Space"/ (co-) Simplicial obj.}$$

$$\hat{\mu}_A(\omega_A) = \sum_{\omega_B \in \Omega_B} \hat{\mu}(\omega_A, \omega_B) \qquad \mu_\phi \leftarrow \mu_A \boxplus \mu_B \leftleftarrows \mu_{AB}$$

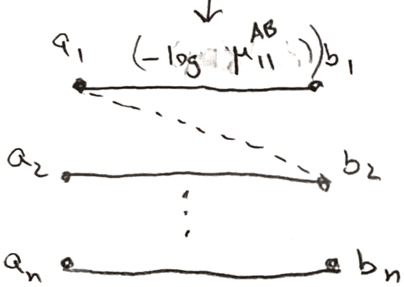
$$\begin{tikzcd}
\text{Multipartite Measures} \arrow[r] \arrow[d, bend right, "\text{Mutual Information}"'] \arrow[dr, bend left] & \boxed{\text{“Spaces” (Simplicial measures)}} \arrow[d, "\text{Homology}"] \\
 & \text{Graded Vector Spaces (+ extra data)} \arrow[d, squiggly, "(\text{“Euler Char”})"] \arrow[dd, bend left, "\text{Euler Char}"] \\
\mathbb{R} & \text{Holomorphic Functions of } q \arrow[l, "\frac{d}{dq}\big|_{q=1}"'] \arrow[d, dashed, "q\to 0"] \\
 & \mathbb{Z}
\end{tikzcd}$$

$\mu : \Omega_A \times \Omega_B \to \mathbb{R}_{\geq 0}$

Probability Matrix (bipartite)

$$\begin{array}{c|cccc}
 & a_1 & a_2 & \cdots & \\
\hline
b_1 & \mu_{11} & & & \\
b_2 & 0 & \mu_{22} & & \\
\vdots & & & \mu_{33} & \\
 & & & & \ddots
\end{array}$$

$\updownarrow$



$$\chi^{\text{aug}}_{\mu} = \sum_i {\mu^{AB}_{ii}}^{q} - 1$$

$$H^0 \cong \{ (\alpha, \beta) : \mathrm{Cov}(\alpha, \beta) = |\mathrm{Var}(\alpha)|^2 = |\mathrm{Var}(\beta)|^2 \}$$

$\alpha : \Omega_A \to \mathbb{C}$

$\beta : \Omega_B \to \mathbb{C}$

$$\begin{tikzcd}
 & \text{State index} \in \mathcal{O}(\mathbb{C}_\alpha \times \mathbb{C}_r \times \mathbb{C}_q) \arrow[dl, "\alpha\to 0"', "\frac{1}{r(q-1)}"] \arrow[dr, "q\to 0"] & \\
\text{Tsallis/Rényi deformed MI} \in \mathcal{O}(\mathbb{C}_q \times \mathbb{C}_r) \arrow[d, "r\to 0"'] \arrow[dr, "r\to 1"] & & \in \mathbb{Z} \\
\text{Rényi MI} \arrow[dr, "q\to 1"'] & \text{Tsallis MI} \arrow[d, "q\to 1"] & \\
 & \text{MI} &
\end{tikzcd}$$

6 min

17

Why (should this be possible)?

(• Correlations among Subsystems $\leftrightsquigarrow$ $\boxed{\text{Obstruction}}$ to Factorizability)

"Space / Cohomology"

• Mutual Info. looks like an EC

$$I_{AB} \overset{?}{=} \dim\left[M_A \boxplus M_B \leftleftarrows M_{AB} \right]$$

$$= \underbrace{\dim(M_A)}_{\text{"}S_A\text{?}} + \dim(M_B) - \dim(M_{AB})$$

• μ Factorizes in <u>any</u> way $\Rightarrow$ MI $= 0$
$\not\Leftarrow$

• MI measures info shared among <u>all</u> subsystems

The Category of ~~Measures~~ States

<u>Def</u>: A state is a pair of a $\mathbb{C}$-alg. A and a positive linear map $\mu : A \longrightarrow \mathbb{C}$

$\mu(a^*a) \geq 0 \quad \forall a$

<u>Ex</u>: $A = \prod_{i=1}^{N} M_{n_i}\mathbb{C}$, $\quad \mu : A \longrightarrow \mathbb{C}$

$$a \longmapsto \sum_{i=1}^{N} \mathrm{Tr}_i \left[\hat{\mu}_i \, a \right]$$

$\in (M_n\mathbb{C})_{\geq 0}$

• purely Classical : $n_i = 1 \;\; \forall i \Rightarrow \hat{\mu}_i \in \mathbb{R}_{\geq 0}$

• purely quantum : $N = 1 \;\; (n_1 > 0)$

Fix A an algebra

<u>Def</u>: State_A has

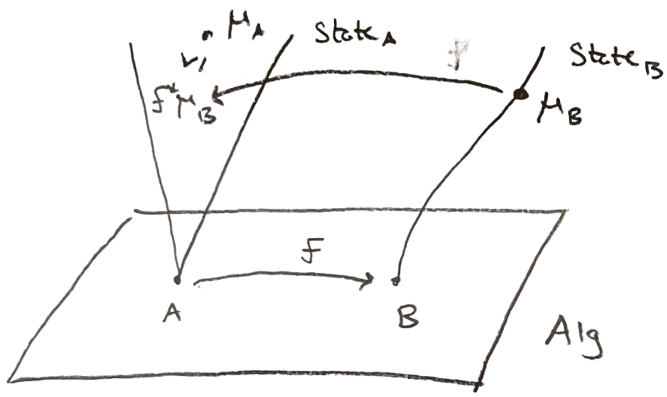
- Objects $\mu : A \to \mathbb{C}$ states
- Morphisms: $\mu \xrightarrow{!} \nu$ if $\nu \le \mu$

$$(\nu(a^*a) \le \mu(a^*a) \ \forall a)$$

$$(\text{RN deriv } \nu = f \cdot \mu)$$

$$\text{“} L^1(X, \mu)_{+ \le} \text{”}$$

State is the (Grothendieck op-Fibration)



$\rightsquigarrow$ $\left(\begin{array}{l} f_\wedge : \mu_B \longrightarrow \mu_A \\ \text{is given by} \\ f : A \to B \quad \text{s.t.} \\ \quad f^*\mu_B \le \mu_A \end{array} \right)$

(Isos are “unitary maps” $u : A \to B$ s.t. $u^*\mu_B = \mu_A$)
$\uparrow$
$*$-isos

- State has coproducts:

$$\mu_A \boxplus \mu_B : A \times B \longrightarrow \mathbb{C}$$
$$(a, b) \longmapsto \mu_A(a) + \mu_B(b)$$

- State has $\otimes$

$$\mu_A \otimes \mu_B : A \otimes B \longrightarrow \mathbb{C}$$
$$a \otimes b \longmapsto \mu_A(a)\mu_B(b)$$

Dimension

$\mathcal{C}$ a category w/ Coproducts and $\otimes$:

$$\dim : \{\text{Iso classes of } \mathcal{C}\} \longrightarrow R \leftarrow \text{ring}$$

$$\dim(C_1 \amalg C_2) = \dim(C_1) + \dim(C_2)$$

$$\dim(C_1 \otimes C_2) = \dim(C_1)\dim(C_2)$$

Ex:

$\mathcal{C}$	dim
Set	Card
$\text{Vect}_{\mathbb{R}}$	$\dim_{\mathbb{R}}$
$(V, f: V \to V)$	$Tr(f^n)$, $n \in \mathbb{Z}_{\geq 0}$
Graded V.S. $V = \bigoplus_k V^k$	$\sum_k (-1)^n \dim_{\mathbb{R}}(V^k)$

- $\mathcal{C}$ = FinMeas $\leftarrow$ States on $\mathbb{C}^n$ / $\hat{\mu}: \Omega \to \mathbb{C}$, $R = \mathcal{O}(\mathbb{C})$

$$\dim_q(\mu) = \sum_i \hat{\mu}_i^{\,q}$$

- $\mathcal{C}$ = Fin State, $R = \mathcal{O}(\mathbb{C}^3)$

$$\dim_{\alpha,q,r}(\mu) = \sum_i n_i^{\alpha}\, Tr[\hat{\mu}_i^{\,q}]^r$$

(EC of $\mu_\phi \leftarrow \mu$)

This inspires

$$S^{TR}_{q,r}(\mu) = \frac{1}{q-1}\left[1 - \dim_{0,q,r}(\mu)\right]$$

$q \to 0$ limit: • $\dim \in \mathbb{Z}$

"L^0 norm"

- Classical case: dim = # points w/ $\mu \neq 0$

Thus, we expect an association:

$$\text{State} \rightsquigarrow \text{Vector Space (or Set)}$$

GNS Modules

- There is a Functor:

$$\mathrm{GNS} : \mathrm{State}_A \longrightarrow {}_A\mathrm{Mod}$$

$$\mu \longmapsto A/\mathcal{I}_\mu \;,\quad \mathcal{I}_\mu = \{ a : \mu(a^* a) = 0 \}$$

$$(\mu \longrightarrow \nu) \longmapsto Q : A/\mathcal{I}_\mu \longrightarrow A/\mathcal{I}_\nu \quad \leftarrow \text{"quotient" map}$$

- $\mathrm{GNS}(\hat{\mu} : \Omega \longrightarrow \mathbb{R}_{\geq 0}) \cong \mathbb{C}[\Omega_{\mu \neq 0}]$ "RN-deriv"

- q Completions:

$$\| - \|_q : \mathrm{GNS}(\mu) \longrightarrow \mathbb{R}_{\geq 0}$$

$$[a] \longmapsto \left| \mu\left((a^* a)^{q/2} \right) \right|^{1/\mathrm{Re}\, q}$$

$\rightsquigarrow$ L^q Spaces

Further Work:

$$\begin{tikzcd}
\mathrm{MPF} \arrow[rrrr, shift right] \arrow[ddrr, shift left] & & & & W^*\mathrm{BMOD}^{\bullet} \arrow[llll, shift right] \arrow[ddll, shift right] \\
\\
& & \mathrm{Stine} \arrow[uull, shift left] \arrow[uurr, shift right] & & \\
\mathrm{FMPF} \arrow[uuu] \arrow[dr, leftrightarrow, "\sim"] \arrow[ddrr, leftrightarrow, "\cong"'] & & & & W^*\mathrm{BMOD}^{\odot} \arrow[uuu] \\
& & \mathrm{MStine} \arrow[uu] \arrow[urr, leftrightarrow, "\sim"] & & \\
& & \mathrm{CP} \arrow[uurr, "\sim"'] & &
\end{tikzcd}$$

$$\boxed{\mathrm{State}^{CP} \cong CP/\mathbb{C}}$$

CP morphisms between algs

- G-equivariant entropy
- Link invariants
- Categorify weird inequalities for entropy / Modular Flow
- Entropy for Finite Fields
- Emergent Geometry / Spacetime