On characterizing classical and quantum entropy

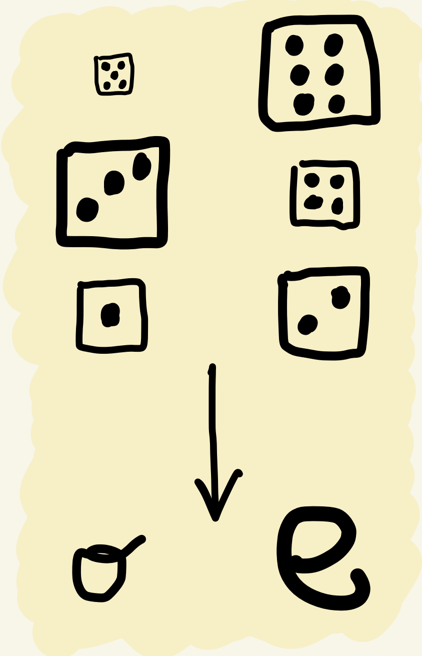
Arthur Parzygnat IHÉS

Categorical semantics of Entropy

CUNY Graduate Center ITS

5/13/2022

Information Loss



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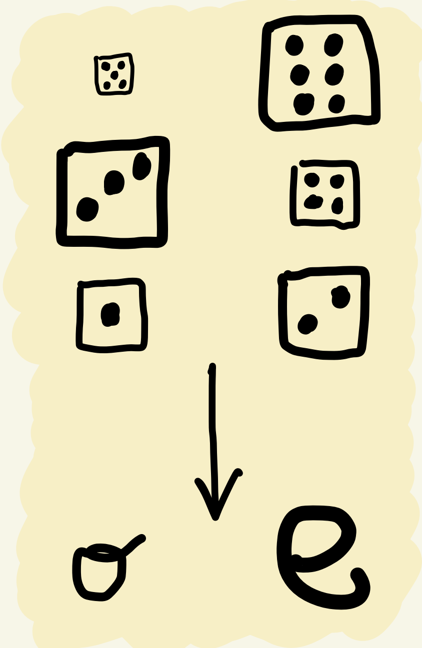

$$\begin{array}{c} (X, p) \\ \Big\downarrow f \\ (Y, q) \end{array}$$

Every (non-injective) function of probability spaces loses information. The entropy difference

$$\Delta_H(f) := H(p) - H(q), \text{ where}$$

$$H(p) := -\sum_x p_x \log(p_x)$$

is the Shannon entropy, quantifies this.

Here, $q_y = \sum_{x \in f^{-1}(\{y\})} p_x$ is the pushforward of p along f, denoted $f \circ p$.

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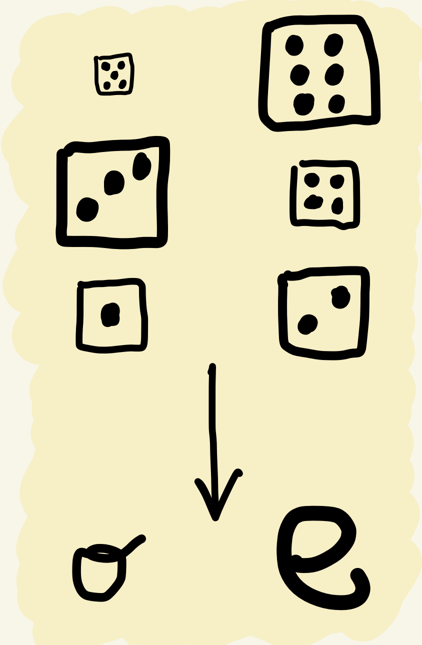

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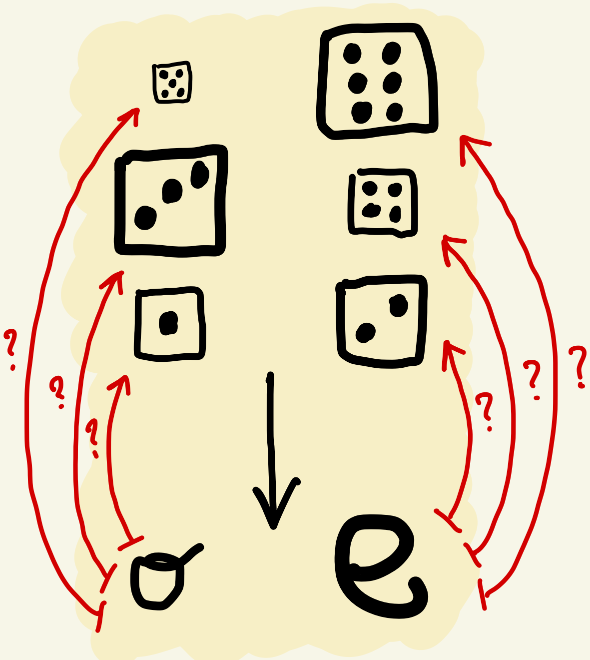
Baez, Fritz, Leinster characterized Δ_H as a continuous convex functor into $\mathbb{B}\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{\geq 0}$ viewed as a one-object category) up to a constant ≥ 0.

Information Recovery



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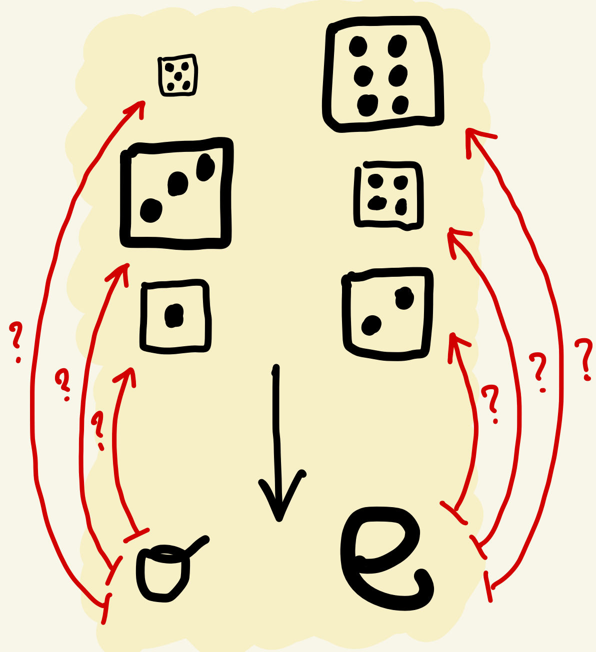


It is impossible to recover information in a deterministic manner!

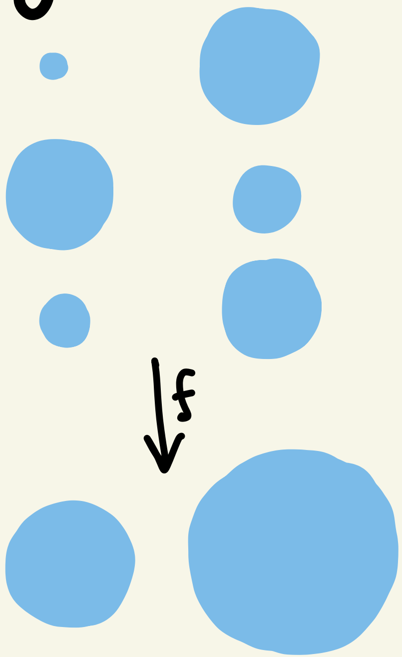
But we can try using a stochastic procedure $Y \overset{h}{\rightsquigarrow} X$.

Notation

h assigns a probability distribution h_y on X to each $y \in Y$.
The value on $x \in X$ is written h_{xy}.

Compare to $\mathbb{P}(x \mid y)$ the conditional probability of x given y.

Hypotheses = stochastic sections



Water droplet picture on left due to Gromov

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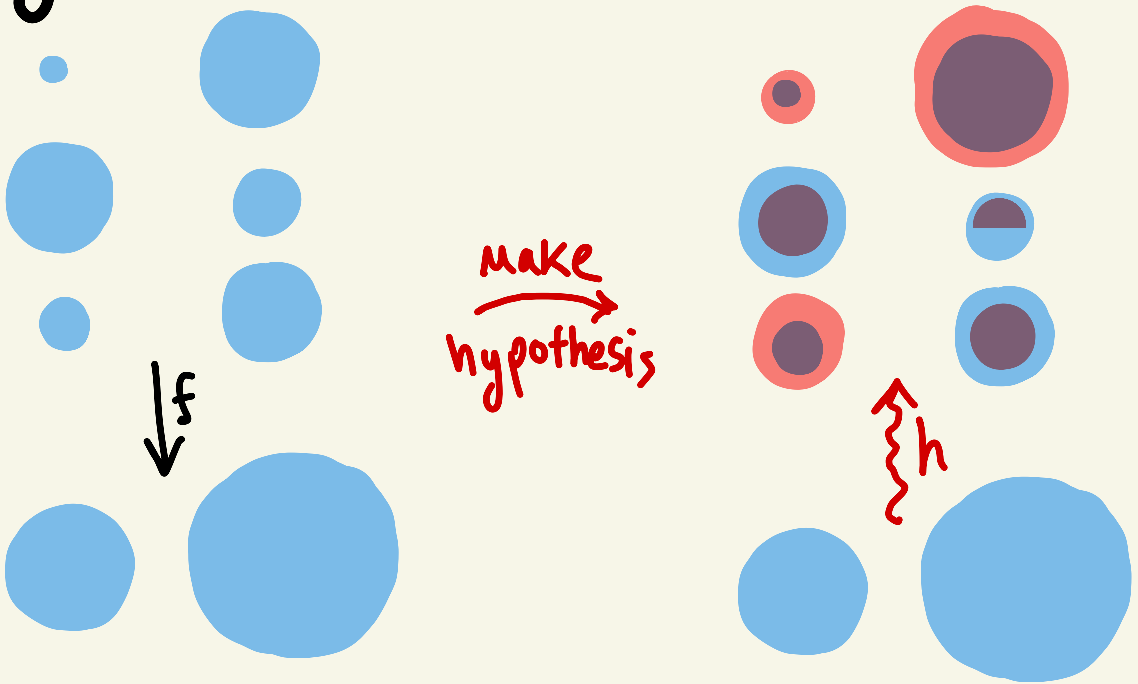

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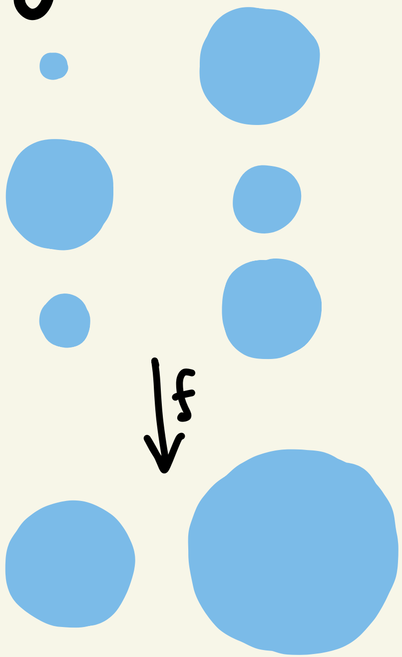
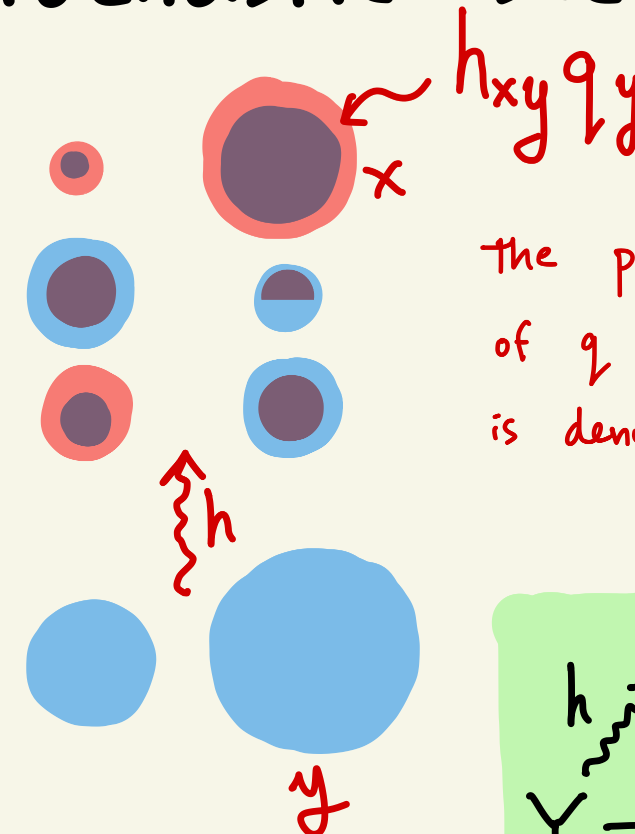


$\xrightarrow{\text{make}}$ hypothesis



the pushforward of q along h is denoted $h \circ q$.

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, squiggly, "h"] \arrow[rr, "\mathrm{id}_Y"'] \arrow[rr, phantom, bend left=20, "="] & & Y \end{tikzcd}$$

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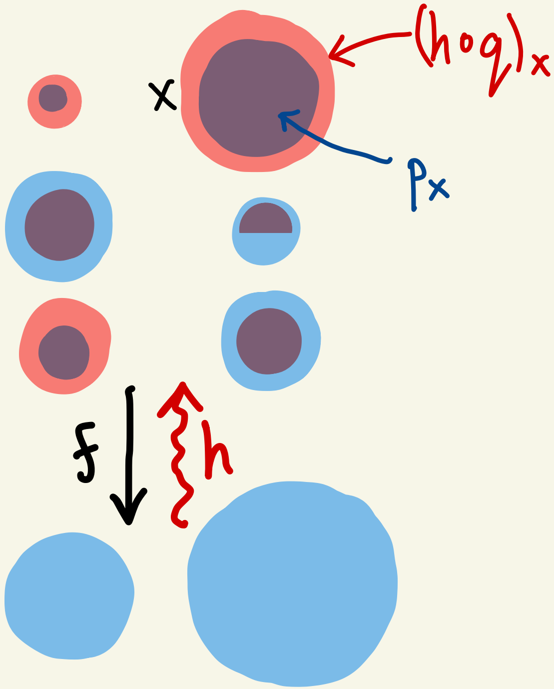
Relative Entropy of Recovery



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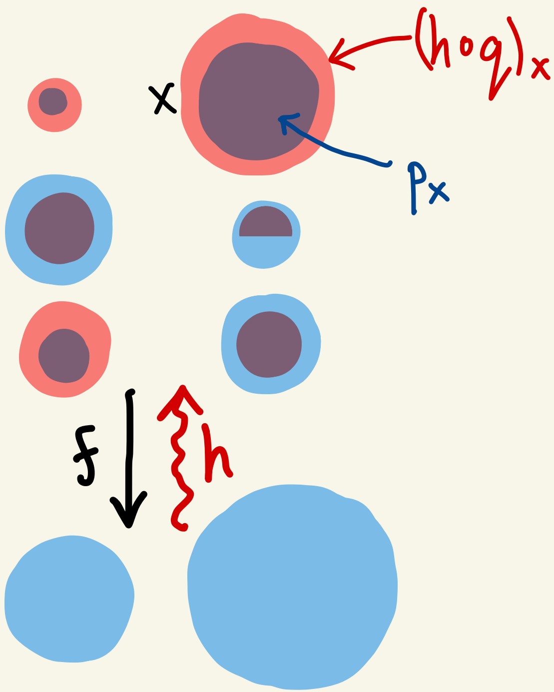

But hypotheses are not always correct. To quantify the accuracy of a hypothesis h, we use the relative entropy (of recovery)

$$S(p\,\|\,h\circ q) := \sum_x p_x \log\left(\frac{p_x}{(h\circ q)_x}\right)$$

$$= \sum_x p_x \log\left(\frac{p_x}{h_{x f(x)}\, q_{f(x)}}\right)$$

h is sometimes called a recovery map

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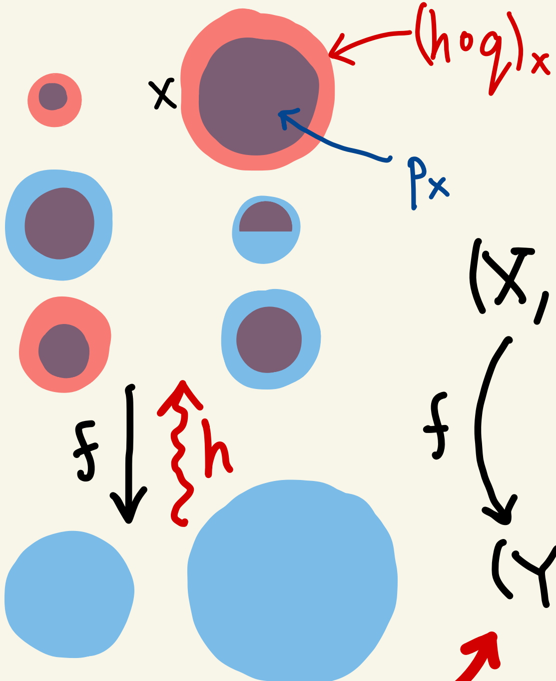

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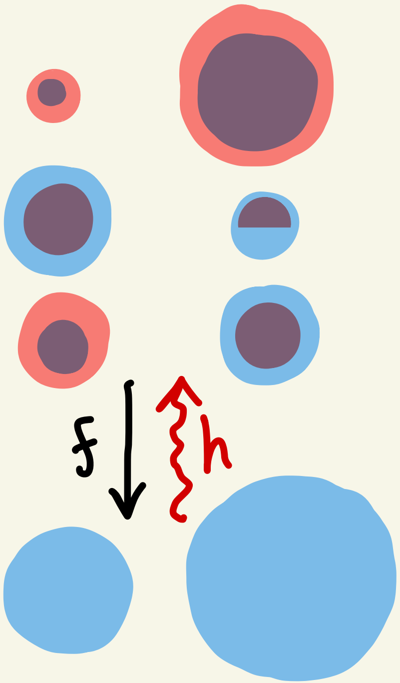
Warning: This notation is misleading.

we require $q = f \circ p$ but <u>NOT</u> $p = h \circ q$!

Relative Entropy & Optimal Hypotheses



The relative entropy satisfies

- $S(p \| h \circ q) \geq 0$

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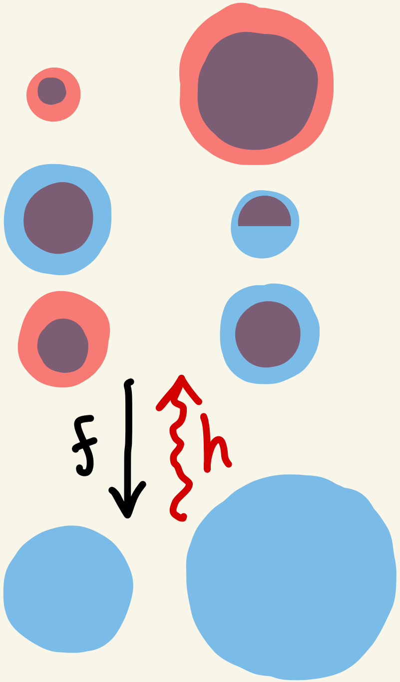

The relative entropy satisfies

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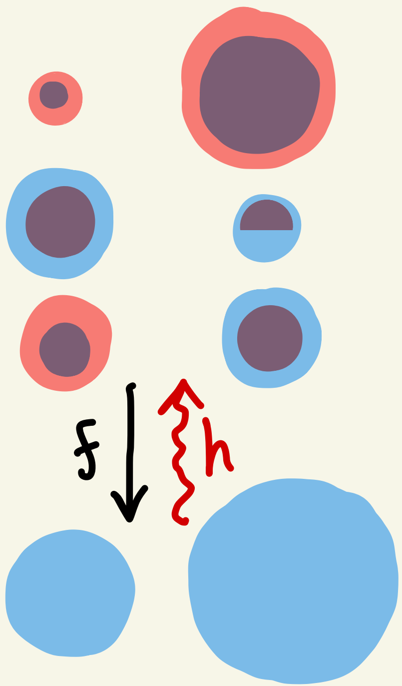

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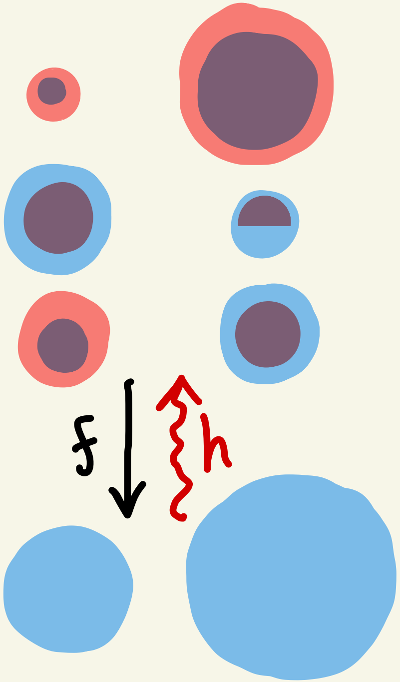

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<u>Fact</u> Optimal hypotheses always exist.

$$h_{xy} = \frac{\delta_{y f(x)} \, p_x}{q_y}, \quad \text{where } \delta_{x'x} = \begin{cases} 1 & \text{if } x' = x \\ 0 & \text{o.w.} \end{cases}$$

A Category of Hypotheses

The following category **FinStat** was constructed by Baez and Fritz.

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$q = f \circ p$

$f \circ h = \mathrm{id}_Y$

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(BUT $\underline{\text{NOT}}$ $p = h \circ q$)

composition:
$$\begin{tikzcd}
(X,p) \arrow[r, bend left, "f"] \arrow[rr, bend left=60, "g \circ f"] & (Y,q) \arrow[l, squiggly, bend left, "h"] \arrow[r, bend left, "g"] & (Z,r) \arrow[l, squiggly, bend left, "k"] \arrow[ll, squiggly, bend left=60, "h \circ k"]
\end{tikzcd}$$

$$(h \circ k)_{xz} := \sum_y h_{xy} k_{yz}$$

Chapman–Kolmogorov

Relative Entropy as a Functor

$$\mathrm{FinStat} \xrightarrow{RE} \mathbb{B}[0,\infty]$$

$\mathbb{B}[0,\infty]$ is the category with a single object and the morphism set is $[0,\infty]$ with addition as the composition.

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$\mathbb{B}[0,\infty]$ is the category with a single object and the morphism set is $[0,\infty]$ with addition as the composition.

is the unique lower semi-continuous convex functor that vanishes on the subcategory FP of optimal hypotheses (up to a non-negative constant).

This is the main theorem of Baez and Fritz.

Functoriality of Relative Entropy

$$
\begin{tikzcd}
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Functoriality of Relative Entropy

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(X,p) \arrow[r, bend left, "f"] \arrow[rr, bend left=50, "g\circ f"] & (Y,q) \arrow[l, squiggly, bend left, "h"] \arrow[r, bend left, "g"] & (Z,r) \arrow[l, squiggly, bend left, "k"] \arrow[ll, squiggly, bend left=50, "h\circ k"]
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Special case of functoriality 1:

$$
\begin{tikzcd}
(X,q) \arrow[r, bend left, "f"] \arrow[rr, bend left=50, "!_X"] & (Y,f\circ q) \arrow[l, squiggly, bend left, "g"] \arrow[r, bend left, "!_Y"] & (\bullet,1) \arrow[l, squiggly, bend left, "f\circ p"] \arrow[ll, squiggly, bend left=50, "p"]
\end{tikzcd}
$$

f ← fixed

g ← optimal hypothesis of (f,p)

p ← fixed

arbitrary → q

Functoriality of Relative Entropy

$$
\begin{tikzcd}
(X,p) \arrow[r, "f"] \arrow[rr, bend left=40, "g\circ f"] & (Y,q) \arrow[r, "g"] \arrow[l, squiggly, bend left, "h"] & (Z,r) \arrow[l, squiggly, bend left, "K"] \arrow[ll, squiggly, bend left=40, "h\circ K"]
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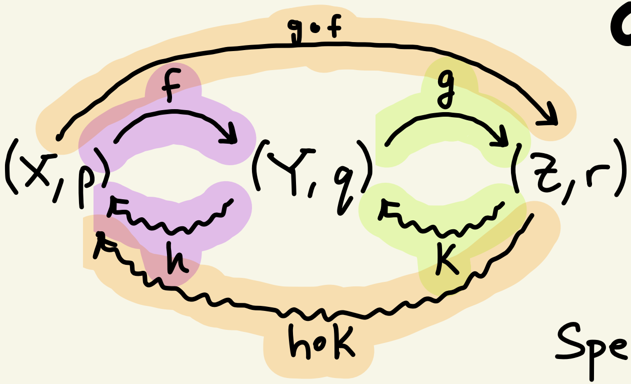
arbitrary

$$S(q \,\|\, g\circ f\circ q) + S(f\circ q \,\|\, f\circ p) = S(q \,\|\, p) \quad \text{i.e.,}$$

$$S(q \,\|\, p) - S(f\circ q \,\|\, f\circ p) = S(q \,\|\, g\circ f\circ q)$$

Functoriality of Relative Entropy



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The conditional expectation property

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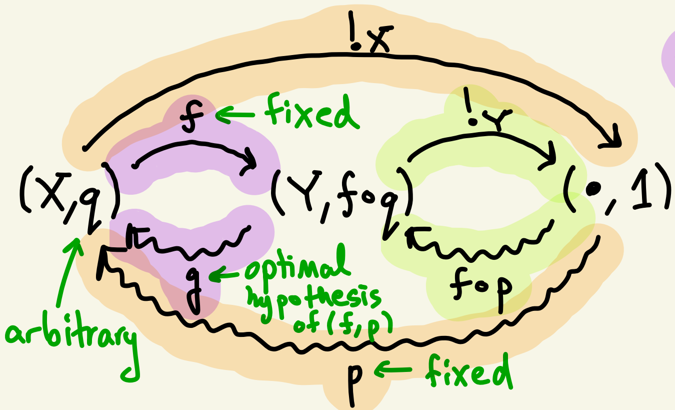
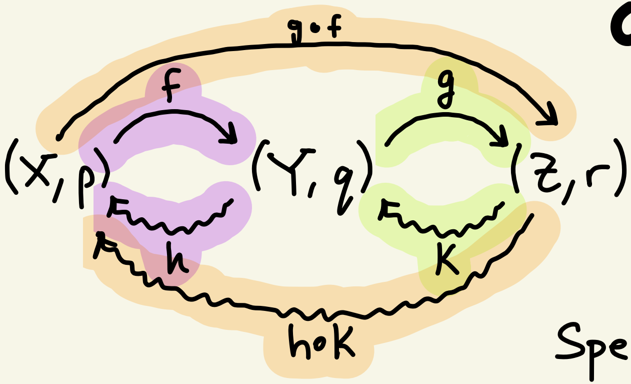


Functoriality of Relative Entropy



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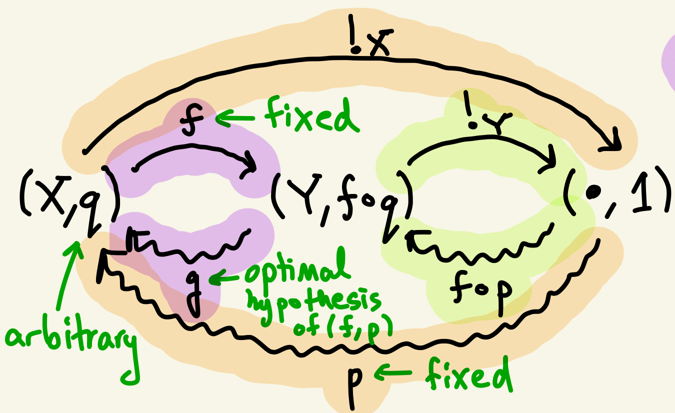
$$\underbrace{S(q\,\|\,p) - S(f\circ q\,\|\,f\circ p)} = S(q\,\|\,g\circ f\circ q)$$

This term appears in the data-processing inequality (DPI). Since the RHS ≥ 0, this is an improvement of the DPI.



Functoriality of Relative Entropy

Special case of functoriality 2:

$$
\begin{tikzcd}
(X \times Y \times Z, p_{XYZ}) \arrow[r, bend left, "\pi^{XYZ}_{YZ}"] \arrow[rr, bend left=60, "\pi^{XYZ}_{Z}"] & (Y \times Z, p_{YZ}) \arrow[r, bend left, "\pi^{YZ}_{Z}"] \arrow[l, squiggly, bend left, "u^{YZ}_{XYZ}"] & (Z, p_Z) \arrow[l, squiggly, bend left, "u^{Z}_{YZ}"] \arrow[ll, squiggly, bend left=60, "u^{Z}_{XYZ}"]
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The π's are projections, while the u's are fiberwise uniform distributions.

Functoriality of Relative Entropy

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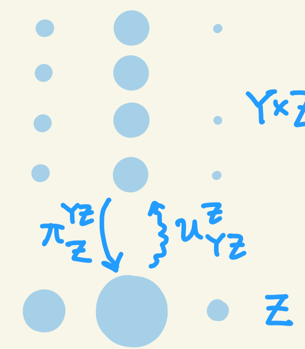

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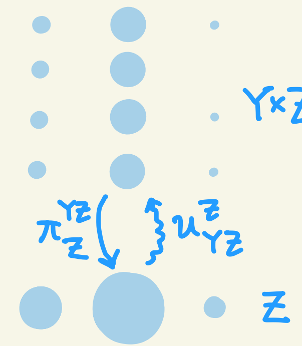

$$S(p_{XYZ} \,\|\, u^{YZ}_{XYZ} \circ p_{YZ}) + S(p_{YZ} \,\|\, u^{Z}_{YZ} \circ p_Z) = S(p_{XYZ} \,\|\, u^{XYZ}_{Z} \circ p_Z)$$

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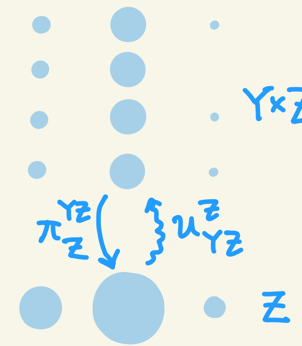

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$$-H(X|YZ) + \log|X| \quad - H(Y|Z) + \log|Y| \quad = \quad -H(XY|Z) + \log|X\times Y|$$

Functoriality of Relative Entropy

Special case of functoriality 2: chain rule for conditional entropy

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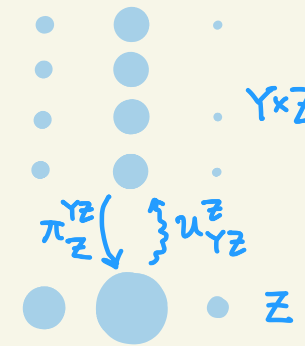

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$$H(X|YZ) + H(Y|Z) = H(XY|Z)$$

[Ref Cover-Thomas Eqn (2.21)]

A stronger DPI

In our first example of functoriality, we found "given $X \xrightarrow{f} Y$ and $\bullet \overset{p}{\rightsquigarrow} X$, set $X \overset{g}{\leftsquigarrow} Y$ to be the optimal hypothesis, then $S(q\|p) - S(f\circ q\|f\circ p) = S(q\|g\circ f\circ q)$ $\forall q$."

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More generally, what can we say if f is stochastic?

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More generally, what can we say if f is stochastic?

Li and Winter proved (in 2014!)

"given $X \overset{f}{\rightsquigarrow} Y$ and $\bullet \overset{p}{\rightsquigarrow} X$, set $X \overset{g}{\leftsquigarrow} Y$ to be the Bayesian inverse of (f,p), then $S(q\|p) - S(f\circ q\| f\circ p) \geq S(q\|g\circ f\circ q)$ $\forall q$."

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"given $X \xrightarrow{f} Y$ and $\bullet \overset{p}{\rightsquigarrow} X$, set $X \overset{g}{\leftsquigarrow} Y$ to be the optimal hypothesis, then $S(q\|p) - S(f\circ q \| f\circ p) = S(q \| g\circ f\circ q)$ $\forall q$."
More generally, what can we say if f is stochastic?

Li and Winter proved (in 2014!)
"given $X \overset{f}{\rightsquigarrow} Y$ and $\bullet \overset{p}{\rightsquigarrow} X$, set $X \overset{g}{\leftsquigarrow} Y$ to be the Bayesian inverse of (f, p), then $\boxed{S(q\|p) - S(f\circ q \| f\circ p) \geq S(q \| g\circ f\circ q)}$ $\forall q$."
Note that this implies monotonicity of relative entropy

$$S(q\|p) - S(f\circ q \| f\circ p) \geq 0,$$

which is sometimes called the data-processing inequality (DPI).

Bayesian inverses

what's this?

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- If g is an optimal hypothesis, then g is a Bayesian inverse and f is necessarily deterministic.

Enough classical stuff!

Li and Winter are quantum information theorists!

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- adjust Hawking's calculation to prove information conservation in black hole evaporation (formerly "the information paradox")

Classical $\longleftrightarrow$ Quantum^{op}

space X

*-algebra of functions
$\mathbb{C}^X := \{X \xrightarrow{\varphi} \mathbb{C}\}$

*-algebra $\mathcal{A}$

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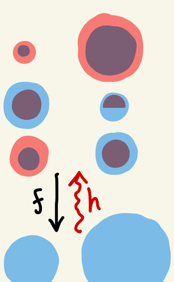

hypothesis

$$\begin{tikzcd} (X, p) \arrow[d, bend right, "f"'] \\ (Y, q) \arrow[u, squiggly, bend right, "h"'] \end{tikzcd}$$

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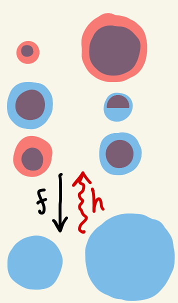

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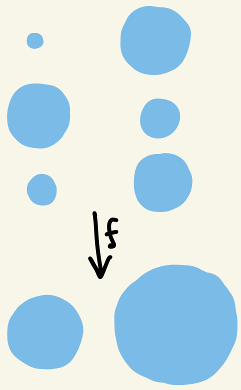
Subtle differences in quantum



Classically, info loss

$$\Delta_H(f) = H(p) - H(q)$$

associated to $(X,p) \xrightarrow{f} (Y,q)$

is always non-negative.

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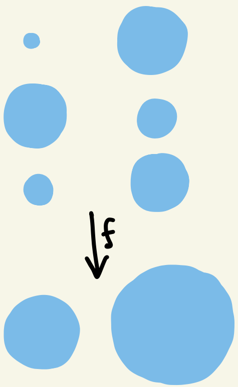

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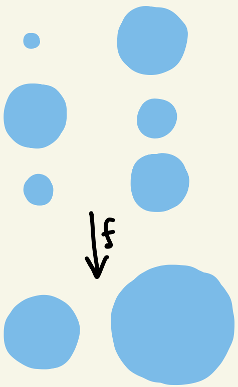

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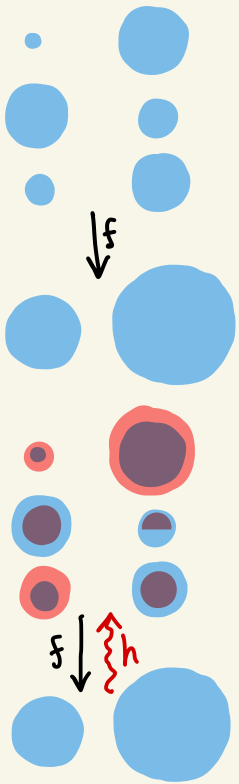

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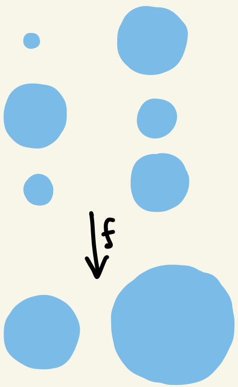

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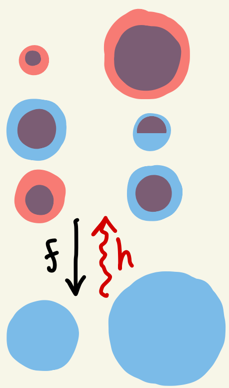

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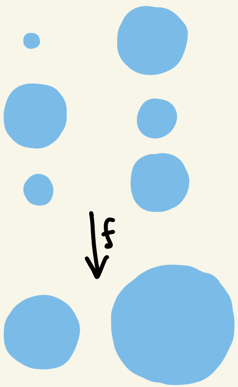

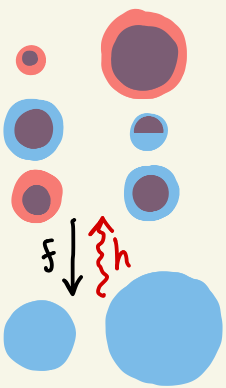

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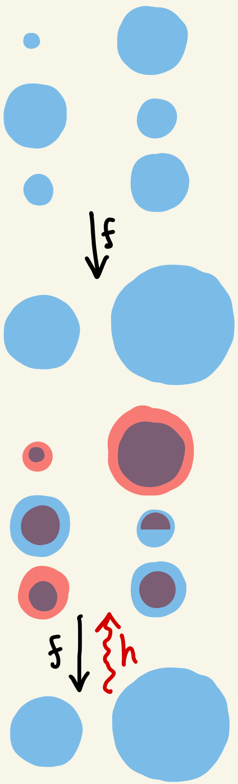

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See my talk @ N. Yanofsky's seminar

This was one of the key observations in extending Baez-Fritz-Leinster's Theorem to quantum.

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Defining NCFinStat (non commutative) via

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$\mathbf{NCFinStat} \xrightarrow{RE} \mathbb{B}[0,\infty]$ also defines a convex functor

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In progress:
1) functoriality for all states
2) lower semi-continuity
3) characterization

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But, the Li-Winter strengthened DPI fails. Namely,

"Given $B \overset{F}{\rightsquigarrow} A$ and $A \overset{\omega}{\rightsquigarrow} \mathbb{C}$, there exists a

$B \overset{G}{\leftsquigarrow} A$ s.t. $S(\xi \| \omega) - S(\xi \circ F \| \omega \circ F) \geq S(\xi \| \xi \circ F \circ G) \ \forall \xi$"

is false.

Properties of RE

Functoriality specializes to

- The conditional expectation property (originally due to Petz)
- The chain rule for quantum conditional entropy

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is false. Variants and weaker versions have been found.

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In progress: Is it true when a Bayesian inverse exists?

Such a map is defined by

$$\begin{tikzcd} \mathcal{A} \arrow[r, squiggly, "\omega"] & \mathbb{C} \arrow[d, phantom, "="] & \mathcal{B} \arrow[l, squiggly, "\omega \circ F"'] \\ \mathcal{A} \otimes \mathcal{A} \arrow[u, squiggly, "\text{mult}"'] & \mathcal{A} \otimes \mathcal{B} \arrow[l, squiggly, "\mathrm{id}_{\mathcal{A}} \otimes F"] \arrow[r, squiggly, "G \otimes \mathrm{id}_{\mathcal{B}}"'] & \mathcal{B} \otimes \mathcal{B} \arrow[u, squiggly, "\text{mult}"] \end{tikzcd}$$

Categorical approaches to entropy

Classical

Quantum

Categorical approaches to entropy

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Shannon entropy
Baez–Fritz–Leinster

Quantum

Categorical approaches to entropy

Classical	Quantum
Shannon entropy	von Neumann entropy
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Categorical approaches to entropy

Classical

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Conditional entropy
Fullwood – Parzygnat

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Categorical approaches to entropy

Classical	Quantum
Shannon entropy Baez–Fritz–Leinster	von Neumann entropy Parzygnat
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Categorical approaches to entropy

Classical

Shannon entropy
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Conditional entropy
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Relative entropy
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Categorical approaches to entropy

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Shannon entropy Baez–Fritz–Leinster	von Neumann entropy Parzygnat
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Infinite-dim'l generalizations unknown

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Classical

Shannon entropy
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Also homological,
operadic,
topos-theoretic
characterizations of some of these exist.

Infinite-dim'l generalizations unknown

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what connects all of these different approaches?

Thank you!

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