Dimers and Grassmannians

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1st Lecture

Outline:

1st Grassmannian and its cluster structure

2nd Twist automorphism its dimer expansion

3rd dimers, surfaces, and spin-structures

1^{st} Lecture:

Grassmannian and its cluster structure

$$Gr_k^n = \left\{ \begin{array}{l} \text{all } k\text{-dimensional} \\ \text{subspaces of } \mathbb{C}^n \end{array} \right\} \ni$$

subspace identified with full rank $k \times n$ matrix $(v_1, \dots, v_n)$ each $v_i \in \mathbb{C}^k$ up to left mult. by $GL_k(\mathbb{C})$

$$= GL_k(\mathbb{C}) \backslash Mat^*_{k,n}(\mathbb{C})$$

quotient space

$$\Longrightarrow \dim Gr_k^n = k(n-k)$$

given k-element subset $I = \{i_1 < \cdots < i_k\} \subset \{1, \dots, n\}$

$[I] = \det(v_{i_1}, \dots, v_{i_k})$ plücker coordinate

homogeneous, i.e. $\det(gv_{i_1}, \dots, gv_{i_k}) = \det(g)[I]$

for any $g \in GL_k(\mathbb{C})$

therefore, get map $Gr_k^n \xhookrightarrow{\mathcal{E}} \mathbb{CP}^{\binom{n}{k}-1}$ plücker embedding

properties: (1) the map $\mathcal{E}$ is one-to-one

(2) the image of $\mathcal{E}$ is the locus of solutions of $\binom{n}{k-1}\binom{n}{k+1}$ homogeneous quadratic polynomials – so called plücker relations

simplest are the <u>short</u> plücker relations:

$$[Lij][Lst] = [Lis][Ljt] + [Lit][Ljs]$$

where $L \subset \{1, \dots, n\}$ is $(k-2)$ element subset

and $i < s < j < t$ indices disjoint from L

inside $\mathbb{C}^{\binom{n}{k}}$ is the <u>affine cone</u> $\widehat{Gr}_k^n$

$$\widehat{Gr}_k^n = \left\{ \eta \in \mathbb{C}^{\binom{n}{k}} \;\middle|\; pr(\eta) \in \operatorname{im} \mathcal{E} \right\} \cup \{0\}$$

where $pr : \mathbb{C}^{\binom{n}{k}} \setminus \{0\} \longrightarrow \mathbb{CP}^{\binom{n}{k}-1}$ is the projection

with $\dim \widehat{Gr}_k^n = k(n-k)+1$ and <u>coordinate ring</u>

$$\mathbb{C}\left[\widehat{Gr}_k^n\right] = \frac{\mathbb{C}\left[[I] : I \subset \{1, \dots, n\} \text{ k-subset}\right]}{\langle \text{plücker relations} \rangle}$$

<u>cluster algebra</u>

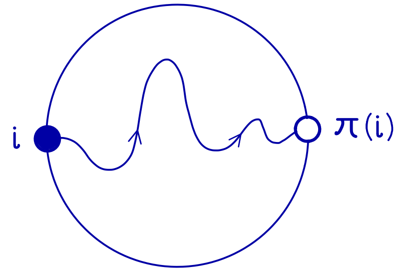
knowing all $\binom{n}{k}$ plücker coordinates is clearly redundant information since $\dim \widehat{Gr}_k^n = k(n-k)+1$ so the initial goal is to construct the plücker clusters, i.e.

- coordinate charts on $\widehat{Gr}_k^n$ each consisting of $k(n-k)+1$ independent plücker coordinates
- transition map between two such charts expressed by Laurent polynomials with positive integer coefficients

Postnikov Diagrams:

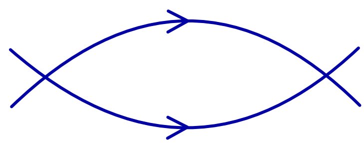
definition: let $\pi \in S_n$ be a permutation; a π-postnikov diagram is a configuration $\mathcal{D}$ of n oriented paths in the disk whose start and end points alternate clockwise along the disk's boundary such that

(1) single path joins source vertex i to sink vertex $\pi(i)$

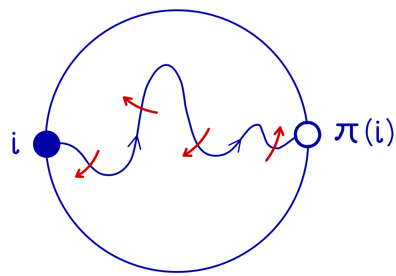


(2) all crossings are <u>transversal</u> and <u>no</u> self-crossings

(3) <u>no</u> unoriented lenses

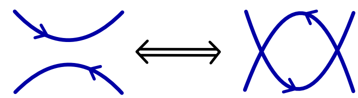


(4) even number of <u>alternating</u> crossings along each path

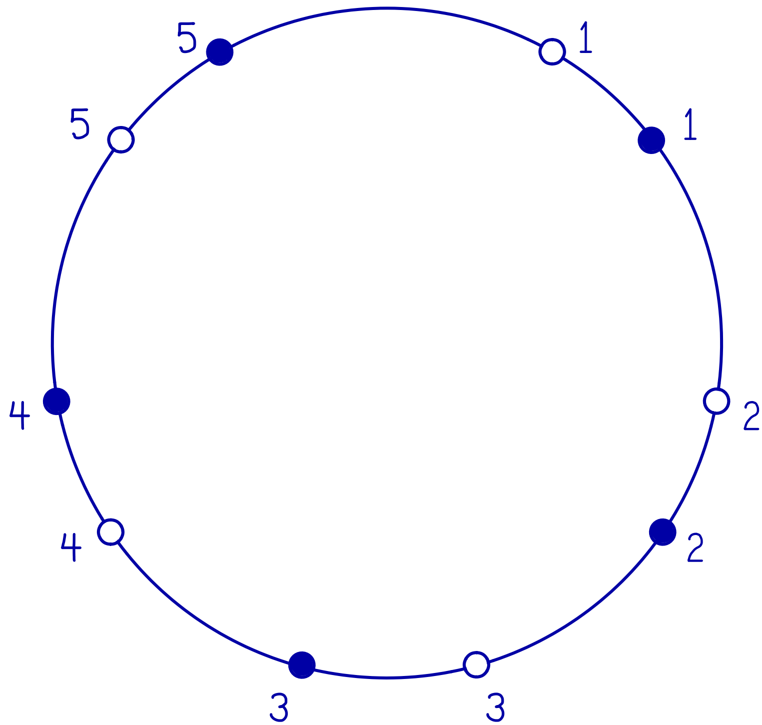


two diagrams considered **equivalent** when related by

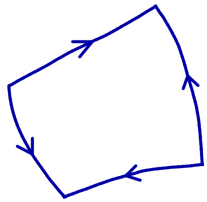
- isotopy
- local creation/annhilation of **oriented lenses**



Example $n = 5$ and $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$



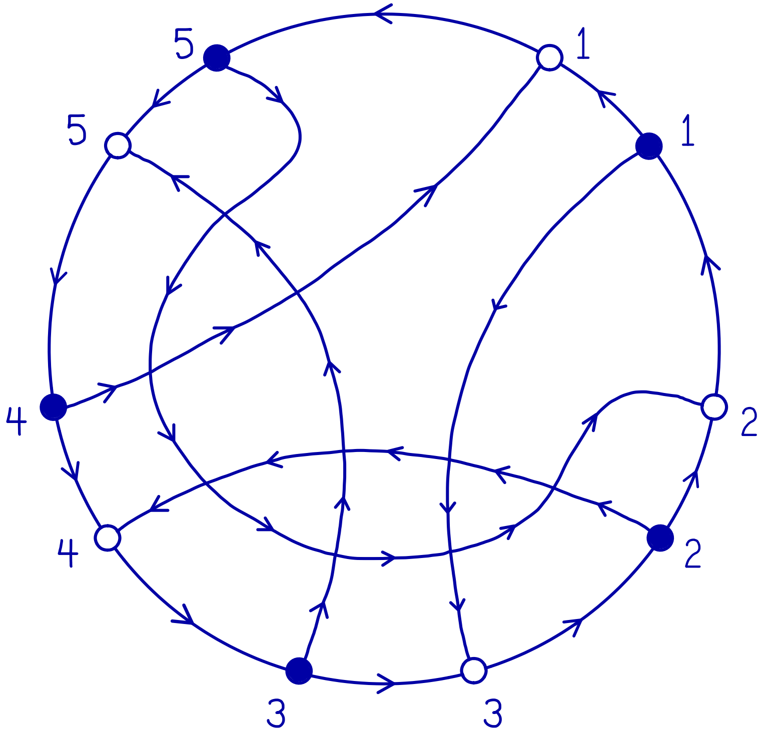
two types of regions in $\mathcal{D}$:



alternating oriented

labeling rule: each alternating region left

the of i^{th} path gets label i



<u>properties</u>: consider the <u>Grassmann permutation</u>
(Postnikov) $\boxed{\pi_{k,n}(i) = (i+k) \bmod n}$ in S_n and let
$\mathcal{D}$ be a $\pi_{k,n}$-postnikov diagram then

(1) # interior alter. regions $= (k-1)(n-k-1)$

boundary alter. regions $= n$ $\boxed{\text{total} = k(n-k)+1}$

(2) each alter. region of $\mathcal{D}$ labeled by a distinct k-element subset $I \subset \{1, \ldots, n\}$

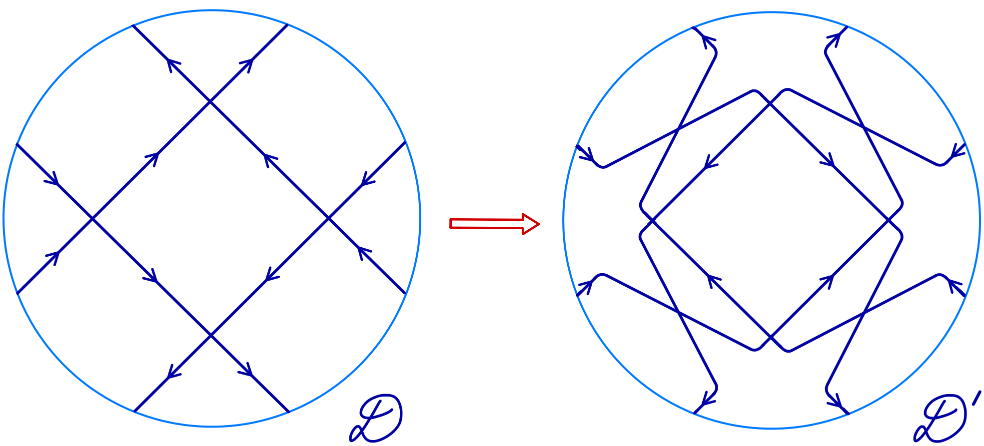
(3) each boundary alter. region labeled by a k-element <u>cyclic interval</u> of $\{1, \ldots, n\}$, i.e. subset of the form $\{(i+r) \bmod n \mid 1 \leqslant r \leqslant k\}$

(4) each k-element subset $I \subset \{1, \ldots, n\}$ occurs as the labeling subset of an alter. region of some $\pi_{k,n}$-postnikov diagram

(5) each pair I, J of labeling subsets of $\mathcal{D}$ is non-crossing, i.e. never find indices $i, j \in I - J$ and $s, t \in J - I$ such that either

$\boxed{i < s < j < t}$ or $\boxed{s < i < t < j}$

<u>definition:</u> the quad-move associated to a interior alter. quadrilateral region of a π-postnikov diagram $\mathcal{D}$ is the following local transformation



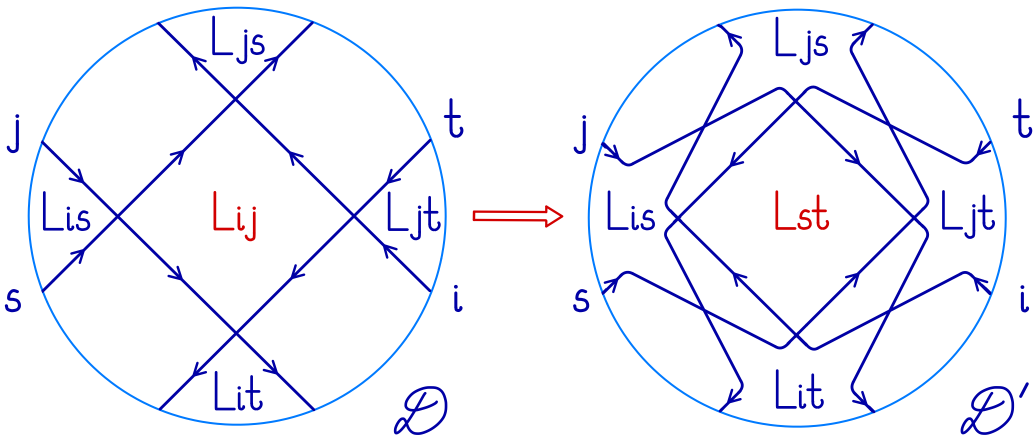
observations (Postnikov):

(1) the result $\mathscr{D}'$ is a π-postnikov diagram

(2) the quad-move is an involution, up to equivalence

(3) any π-postnikov diagram can be obtained from an initial diagram $\mathscr{D}$ by successive quad-moves

(4) the effect of the quad-move on k-subsets is



for some (k−2) element subset $L \subset \{1, \ldots, n\}$ and crossing indices $\{i, j, s, t\}$ disjoint from L

i.e. either $i < s < j < t$ or $s < i < t < j$

<u>remark</u>: L_{ij} is replaced by L_{st} and <u>no</u> other k-subset label of $\mathcal{D}$ changes

<u>remark</u>: the quad-move corresponds to the <u>short plücker relation</u> $[L_{ij}][L_{st}] = [L_{is}][L_{jt}] + [L_{it}][L_{js}]$

<u>definition:</u> the <u>cluster</u> associated to a π-postnikov diagram $\mathcal{D}$ is the set of plücker coordinates

$$\mathbb{X}_{\mathcal{D}} = \left\{ [I] \;\middle|\; \begin{array}{l} \text{k-subset labels } I \text{ of} \\ \text{alter. regions of } \mathcal{D} \end{array} \right\}$$

<u>result</u>: let $\mathcal{D}$ be a $\pi_{k,n}$-postnikov diagram then

(1) for any k-subset $J \subset \{1, \ldots, n\}$ the plücker coordinate $[J]$ can be expressed as a subtraction-free rational polynomial in $[I] \in \mathbb{X}_{\mathcal{D}}$

(2) the plücker coordinates in $\mathbb{X}_{\mathcal{D}}$ are algebraically independent and $\mathbb{C}(\mathbb{X}_{\mathcal{D}}) = \mathbb{C}(\widehat{Gr}_k^n)$

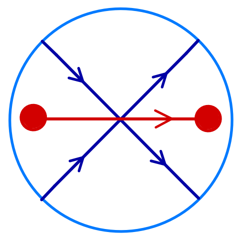
Mutation and Cluster Theory:

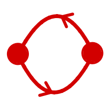
(generalizing quad-moves and short plücker relations)

definition: the quiver associated to a π-postnikov diagram $\mathcal{D}$ is the oriented graph $Q_{\mathcal{D}}$ whose

(1) vertices correspond to the alter. regions of $\mathcal{D}$

(2) an edge is drawn when two alter. regions are incident at an intersection of two paths according to the following local rule

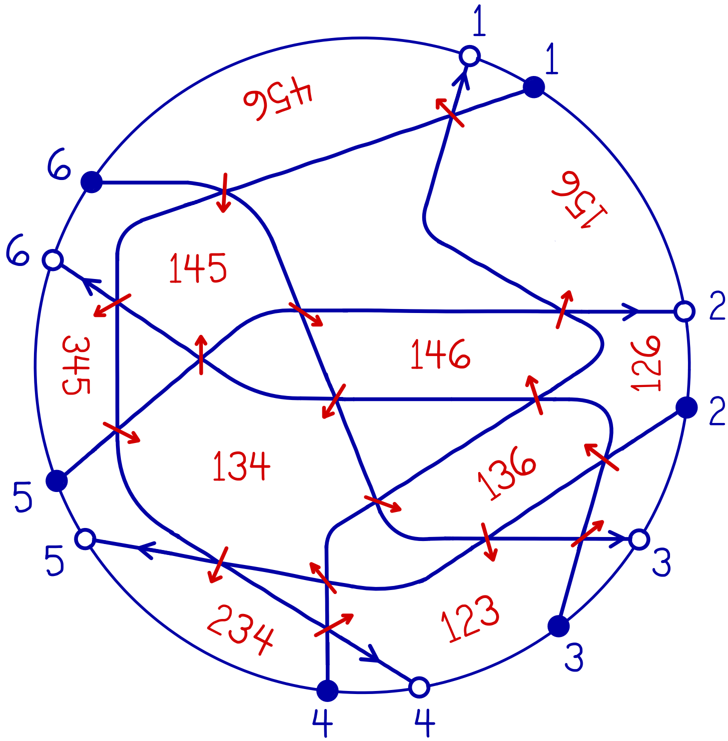


remark: before constructing $Q_{\mathcal{D}}$ we annhilate all local oriented lenses in $\mathcal{D}$ to avoid creating oriented 2-cycles  in the quiver

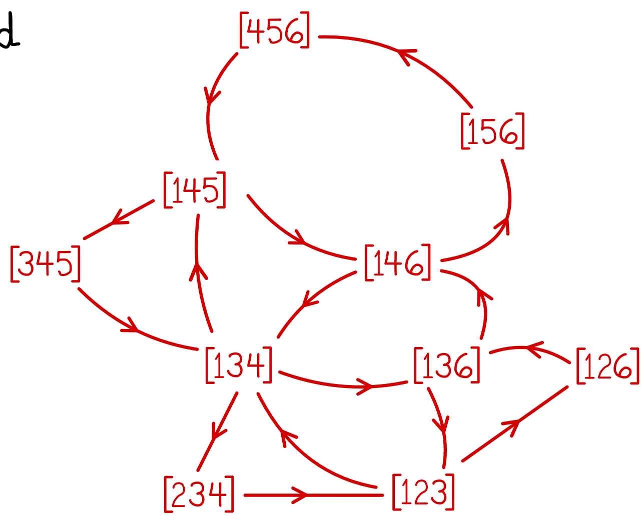
definition: the seed of the π-postnikov diagram $\mathcal{D}$ is the pair $(Q_{\mathcal{D}}, X_{\mathcal{D}})$

Example:

$\pi_{3,6}$ postnikov diagram $\mathcal{D}$



corresponding seed quiver $Q_{\mathcal{D}}$ and cluster $\mathbb{X}_{\mathcal{D}}$



definition: a seed $(Q, \mathcal{X})$ is an oriented graph Q without oriented 2-cycles (i.e. a quiver) along with an algebraically independent set $\mathcal{X} = \{x_v : v \in Q \text{ vertex}\}$ of rational functions (i.e. a cluster)

for example on $\widehat{Gr}_k^n$

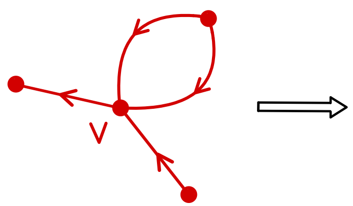
definition: Mutation μ_v of a seed $(Q, \mathcal{X})$ at a vertex $v \in Q$ is a transformation which creates a new seed $(Q', \mathcal{X}')$ where

(1) $\mathcal{X}'$ is obtained by replacing $x_v \in \mathcal{X}$ with x'_v defined by the exchange relation

$$x_v \, x'_v = \prod_{w \xrightarrow{m} v} x_w^m + \prod_{w \xleftarrow{m} v} x_w^m$$

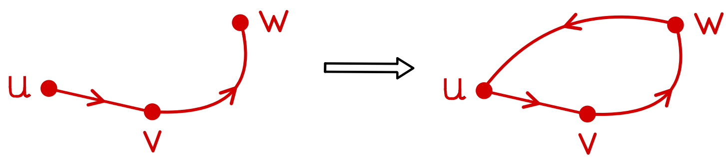
(2) Q' is obtained in three steps by

I. reversing orientations of all edges incident to v

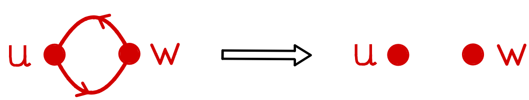


II. create an edge $w \longrightarrow u$ for each pair of edges

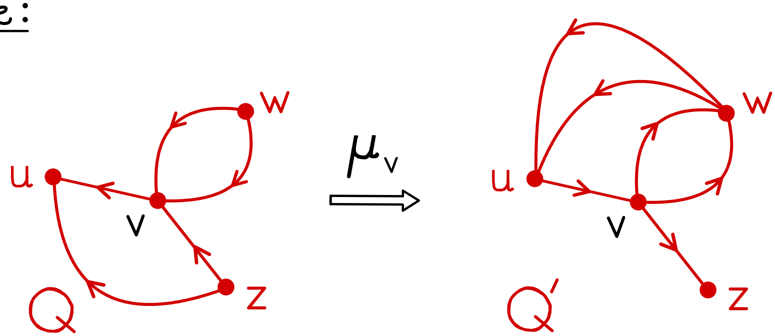
$u \longrightarrow v$ and $v \longrightarrow w$



III. annhilate any oriented 2-cycle created by step II



Example:



exchange relation $\quad x_v x_v' = x_w^2 x_z + x_u$

<u>observations</u> :

(1) $\mathbb{C}(\mathbb{X}) = \mathbb{C}(\mathbb{X}')$ fields of rational polynomials

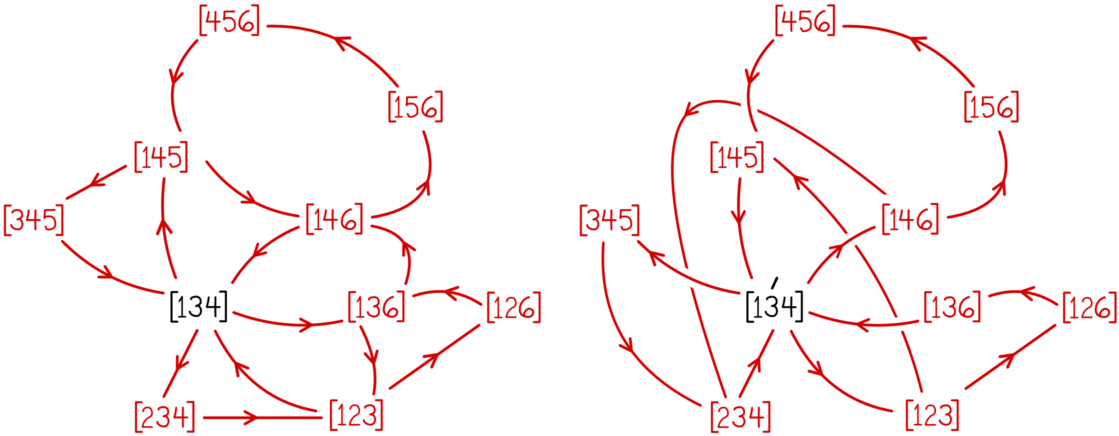
(2) Mutation is an <u>involution</u> ; $\mu_v^2(Q,\mathbb{X}) = (Q,\mathbb{X})$ for any vertex $v \in Q$

<u>remark</u> : each <u>quad</u>-<u>move</u> of $\mathcal{D}$ corresponds to <u>Mutating</u> $Q_{\mathcal{D}}$ at the 4-valent vertex associated to the interior alter. quadrilateral region

<u>remark</u> : in this case the <u>short</u> <u>plücker relation</u> is the <u>exchange relation</u>

<u>remark</u> : the quiver $Q'_{\mathcal{D}}$ obtained by mutating $Q_{\mathcal{D}}$ at a <u>non</u> <u>4-valent</u> vertex <u>no</u> longer corresponds to a π-postnikov diagram

<u>Example:</u>



$(Q_{\mathscr{D}}, \mathbb{X}_{\mathscr{D}})$ $(Q'_{\mathscr{D}}, \mathbb{X}'_{\mathscr{D}})$

the <u>exchange relation</u> in this case reads

$$[134][134]' = [123][345][146] + [234][145][136]$$

and using short plücker relations we get

$$[134]' = [236][145] - [123][456] \in \mathbb{C}[\widehat{Gr}_3^6]$$

$[134]'$ is an example of a <u>twisted</u> plücker coordinate definition: a plücker coordinate $[I]$ is frozen if the k-subset I is a cyclic interval in $\{1, \dots, n\}$

definition: a vertex $v \in Q$ of a seed $(Q, \mathbb{X})$ is frozen if $x_v \in \mathbb{X}$ is a frozen plücker coordinate

all other vertices of Q are active

definition: let $\mathcal{D}$ be any $\pi_{k,n}$-postnikov diagram; a Grassmann seed is a seed $(Q, \mathbb{X})$ obtained by repeatedly mutating $(Q_{\mathcal{D}}, \mathbb{X}_{\mathcal{D}})$ using only active vertices

definition: a cluster variable is any rational function $x \in \mathbb{X}$ which is not a frozen plücker coordinate and where $(Q, \mathbb{X})$ is a Grassmann seed; $\mathcal{X}$ will denote the set of all cluster variables

Theorem: let $x \in \mathcal{X}$ be any cluster variable then

(1) x is a regular function on $\widehat{Gr}_k^n$, i.e. $x \in \mathbb{C}[\widehat{Gr}_k^n]$

(2) x can be uniquely expressed as a Laurent polynomial in $x' \in \mathcal{X}'$ with positive integer coefficients and where $(Q', \mathcal{X}')$ is any Grassmann seed