Dimers and Grassmannians

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2nd Lecture

2^{nd} Lecture:

Twist automorphism its dimer expansion

Notation: for $1 \leqslant i \leqslant n$ let $\mathring{i} = \{(i-r) \bmod n \mid 0 \leqslant r \leqslant k-1\}$ and $\sigma \in S_n$ be the permutation $\sigma(i) = (i-1) \bmod n$

definition: the generalized cross-product $v_1 \times \cdots \times v_{k-1}$ of $v_1, \ldots, v_{k-1} \in \mathbb{C}^k$ is the vector in $\mathbb{C}^k$ satisfying

$$\langle v_1 \times \cdots \times v_{k-1}, w \rangle = \det(v_1, \ldots, v_{k-1}, w)$$

for all $w \in \mathbb{C}^k$ where $\langle v, w \rangle$ is the standard inner product

definition: the twist $\overleftarrow{\mathcal{M}} = (w_1, \ldots, w_n)$ of a $k \times n$ matrix $\mathcal{M} = (v_1, \ldots, v_n)$ is the $k \times n$ matrix where

$$w_i = \varepsilon_i v_{\sigma^{k-1}(i)} \times v_{\sigma^{k-2}(i)} \times \cdots \times v_{\sigma(i)}$$

and where $\varepsilon_i = \begin{cases} (-1)^{i(k-i)} & \text{if } i \leqslant k-1 \\ 1 & \text{if } i \geqslant k \end{cases}$

<u>Examples:</u> let $\mathcal{M} = (v_1, \dots, v_n)$ be a $k \times n$ matrix

(1) $\overleftarrow{\mathcal{M}} = (-v_n, v_1, \dots, v_{n-1})$ for $k=2$

(2) $\overleftarrow{\mathcal{M}} = (v_{n-1} \times v_n, v_n \times v_1, v_1 \times v_2, \dots, v_{n-2} \times v_{n-1})$ for $k=3$

<u>observations</u>:

(1) $\overleftarrow{\mathcal{M}} \in \mathrm{Mat}^*_{k,n}(\mathbb{C})$ whenever $\mathcal{M} \in \mathrm{Mat}^*_{k,n}(\mathbb{C})$

(2) $\overleftarrow{g\mathcal{M}} = \det(g)\, g^{-T} \overleftarrow{\mathcal{M}}$ for any $g \in GL_k(\mathbb{C})$

(3) the twist $\mathcal{M} \mapsto \overleftarrow{\mathcal{M}}$ induces both a birational map $\varphi : Gr_k^n \longrightarrow Gr_k^n$ as well as a regular map $\widehat{\varphi} : \widehat{Gr}_k^n \longrightarrow \widehat{Gr}_k^n$

<u>definition:</u> the <u>twist</u> $\overleftarrow{f}$ of a function $f \in \mathbb{C}[\widehat{Gr}_k^n]$ is the composition $\overleftarrow{f} = f(\widehat{\varphi})$

<u>remark</u>: the map $f \mapsto \overleftarrow{f}$ is an algebra homorphism

$$\mathbb{C}[\widehat{Gr}_k^n] \longrightarrow \mathbb{C}[\widehat{Gr}_k^n]$$

<u>Example:</u> for $k=3$, $n=6$ calculate $\overleftarrow{[246]}$

$$\mathcal{M} = (v_1, v_2, v_3, v_4, v_5, v_6) \in \mathrm{Mat}^{*}_{3,6}(\mathbb{C})$$

$$\overleftarrow{\mathcal{M}} = (v_5 \times v_6, v_6 \times v_1, v_1 \times v_2, v_2 \times v_3, v_3 \times v_4, v_4 \times v_5)$$

$$\begin{aligned}
\overleftarrow{[246]} &= \det(v_6 \times v_1 \quad v_2 \times v_3 \quad v_4 \times v_5) \\
&= \langle v_6 \times v_1 , (v_2 \times v_3) \times (v_4 \times v_5) \rangle \\
&= \det \begin{pmatrix} \langle v_6, v_2 \times v_3 \rangle & \langle v_6, v_4 \times v_5 \rangle \\ \langle v_1, v_2 \times v_3 \rangle & \langle v_1, v_4 \times v_5 \rangle \end{pmatrix} \\
&= \det \begin{pmatrix} [236] & [456] \\ [123] & [145] \end{pmatrix} \\
&= [236][145] - [123][456] \quad \text{cluster variable!}
\end{aligned}$$

<u>fact:</u> the twist $\overleftarrow{x}$ of any cluster variable $x \in \mathcal{X}$ can be expressed as a product $\overleftarrow{x} = my$ where $y \in \mathcal{X}$ is a cluster variable and m is a <u>monomial</u> consisting of <u>frozen</u> plücker coordinates $[\mathring{i}]$ for $1 \leq i \leq n$

Problem: calculate the unique Laurent polynomial expansion of each twisted plücker coordinate $\overleftarrow{[I]}$ with respect to the cluster $\mathbb{X}_{\mathcal{D}}$ associated to any $\pi_{k,n}$-postnikov diagram $\mathcal{D}$

The Bipartite Dual Graph of a Postnikov Diagram:

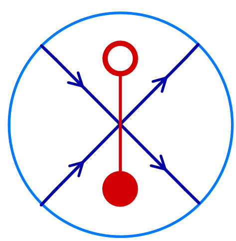
definition: the bipartite graph dual to a π-postnikov diagram $\mathcal{D}$ is the bipartite graph $\Gamma_{\mathcal{D}}$ whose

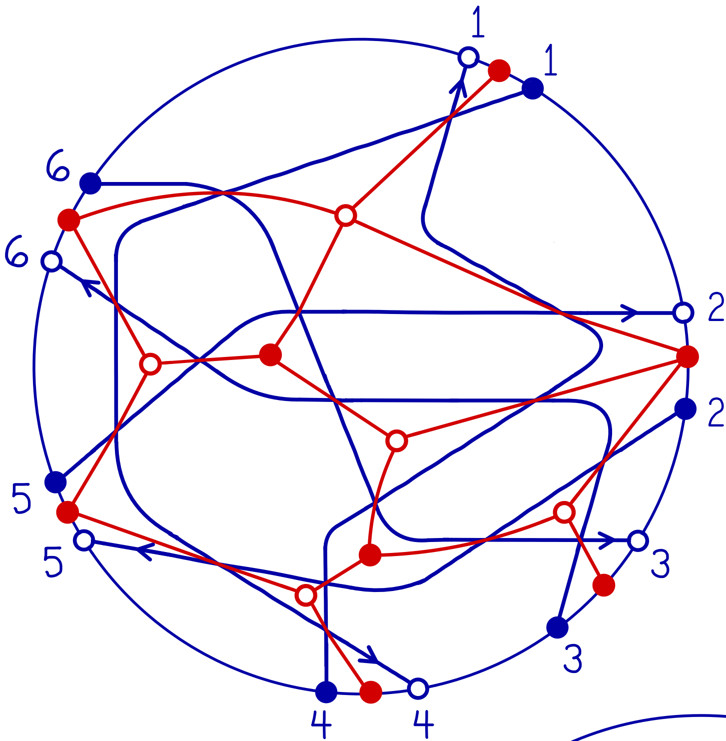
(1) vertices correspond to the oriented regions of $\mathcal{D}$

$\circ$ - vertices $\Longleftrightarrow$ counter-clockwise regions

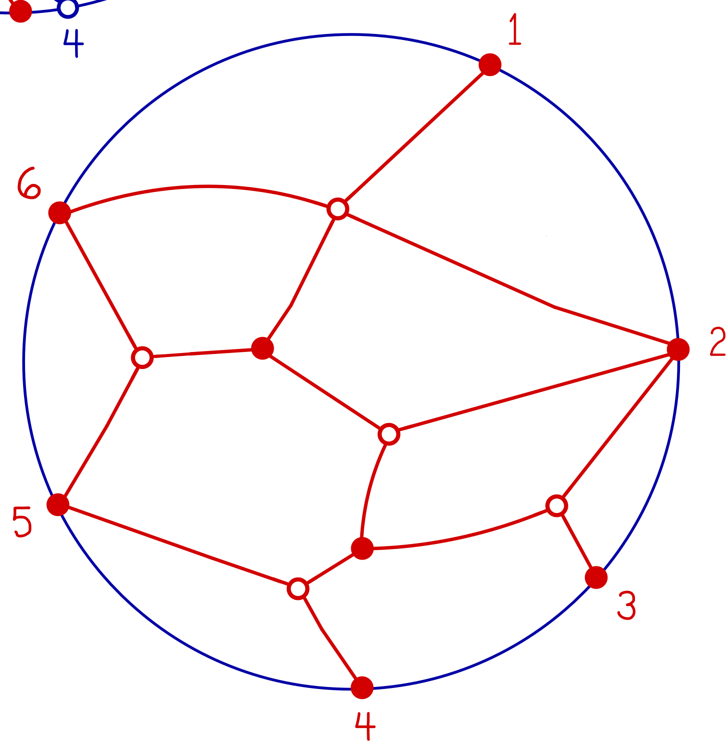
$\bullet$ - vertices $\Longleftrightarrow$ clockwise regions

(2) an edge is drawn when two oriented regions are incident at an intersection of paths





$\pi_{3,6}$ postnikov diagram $\mathcal{D}$



bipartite dual graph $\Gamma_{\mathcal{D}}$

definition: the i^{th} boundary vertex of $\Gamma_{\mathscr{D}}$ is the $\bullet$-vertex which corresponds to the clockwise boundary region of $\mathscr{D}$ touching the i^{th} source and sink, $1 \leqslant i \leqslant n$

definition: let $\partial\Gamma_{\mathscr{D}}$ denote the set of boundary vertices in $\Gamma_{\mathscr{D}}$

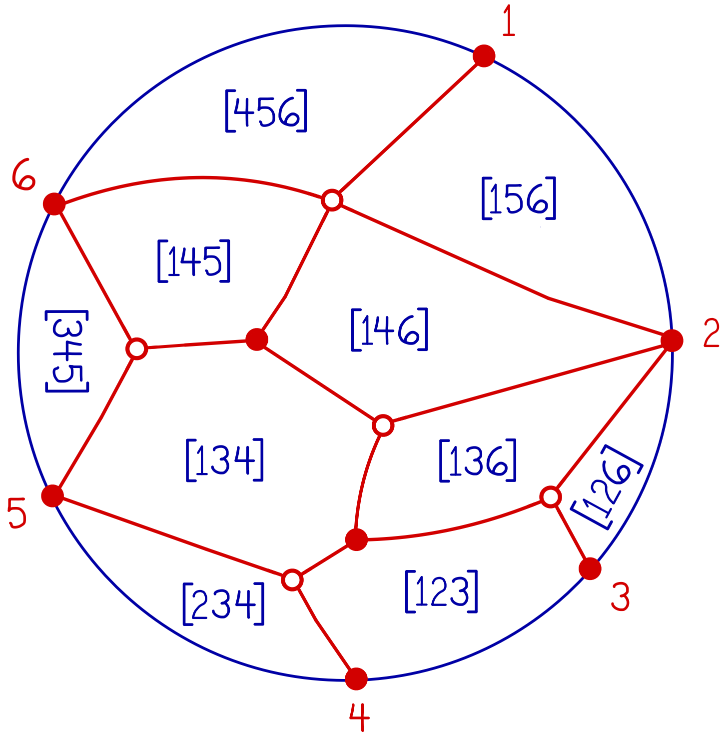
observations:

(1) $\# \bullet$-vertices $= \# \circ$-vertices $+ k$

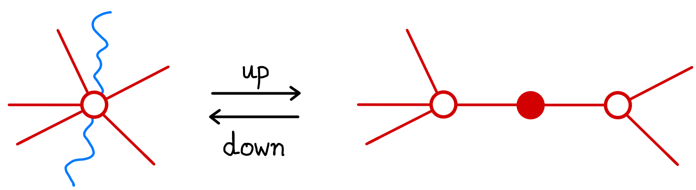
(2) $\# \partial\Gamma_{\mathscr{D}} = n$

remark: the faces of $\Gamma_{\mathscr{D}}$ correspond to the alter. regions of $\mathscr{D}$

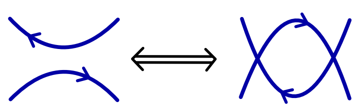
definition: label a face of $\Gamma_{\mathscr{D}}$ by $[I]$ if the corresponding alter. region in $\mathscr{D}$ is labeled by I



two bipartite graphs considered equivalent when related by isotopy and ○-vertex blow-ups and blow-downs



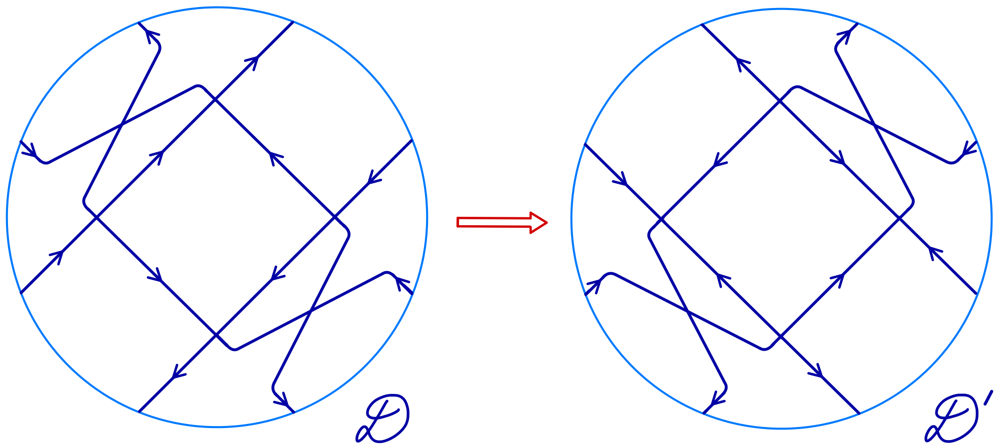
dual version of local creation/annhilation of oriented lenses

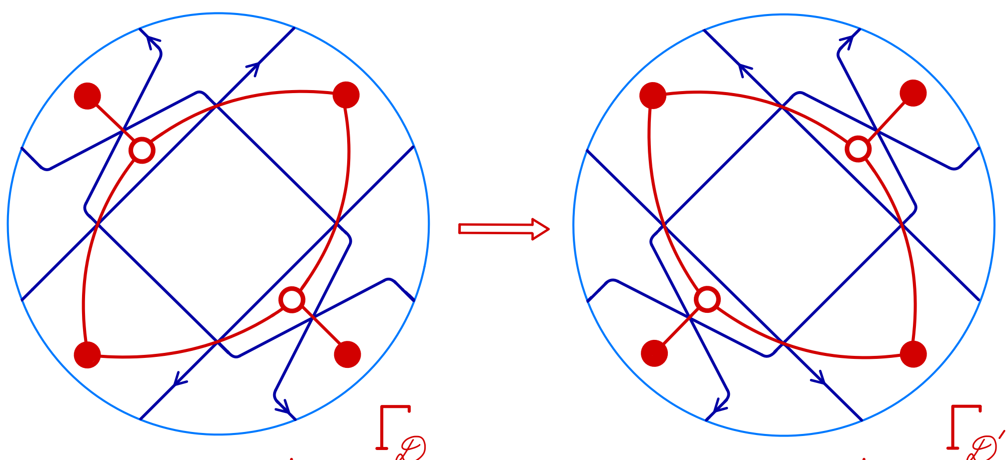


<u>observation:</u> let $\mathscr{D}$ and $\mathscr{D}'$ be π-postnikov diagrams related by a quad-move then the bipartite dual graphs $\Gamma_{\mathscr{D}}$ and $\Gamma_{\mathscr{D}'}$ are related by a <u>spider-move</u>

up to equivalence

up to equivalence

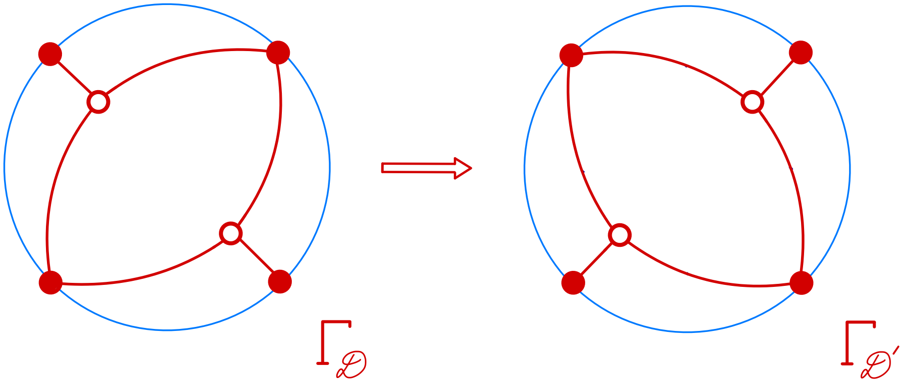




up to equivalence

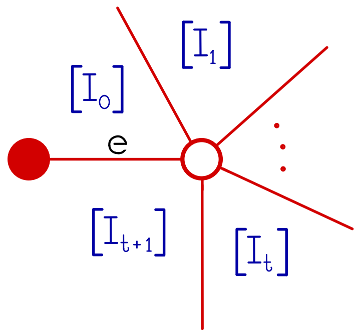
up to equivalence

spider-move



definition: the face-induced edge weight of an edge $e \in \Gamma_{\mathcal{D}}$ is

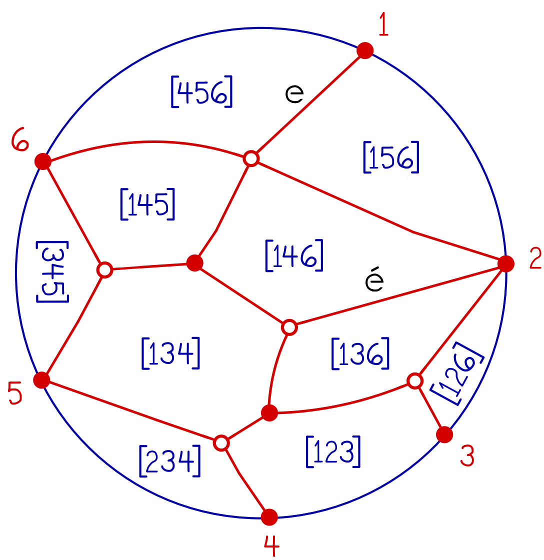
$$\omega(e) = \prod_{s=1}^{t} [I_s] \quad \text{where}$$



Example:

$\omega(e) = [145][146]$

$\omega(é) = [134]$



Dimers:

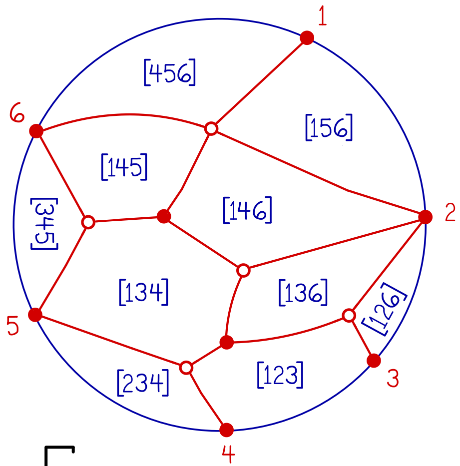
definition: $\Gamma_{\mathcal{D}}^{I}$ will denote the induced subgraph of $\Gamma_{\mathcal{D}}$ obtained by removing the boundary vertices in $\partial\Gamma_{\mathcal{D}}$ labeled by the k-subset $I \subset \{1, \ldots, n\}$

remark: # ●-vertices = # ○-vertices in $\Gamma_{\mathcal{D}}^{I}$

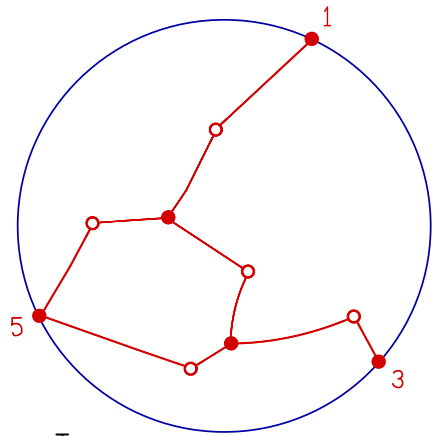
definition: a dimer configuration δ is a subset of edges in $\Gamma_{\mathcal{D}}^{I}$ such that each vertex $v \in \Gamma_{\mathcal{D}}^{I}$ is incident to exactly one edge $e \in \delta$

definition: the dimer partition function of $\Gamma_{\mathcal{D}}^{I}$ is

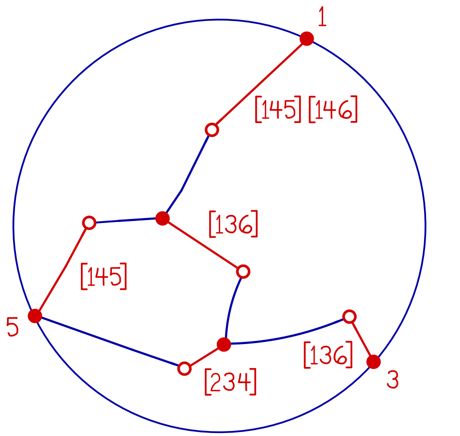
$$\Delta_{\mathcal{D}}^{I} = \sum_{\text{dimers } \delta \text{ on } \Gamma_{\mathcal{D}}^{I}} \omega(\delta) \quad \text{with} \quad \omega(\delta) = \prod_{e \in \delta} \omega(e)$$

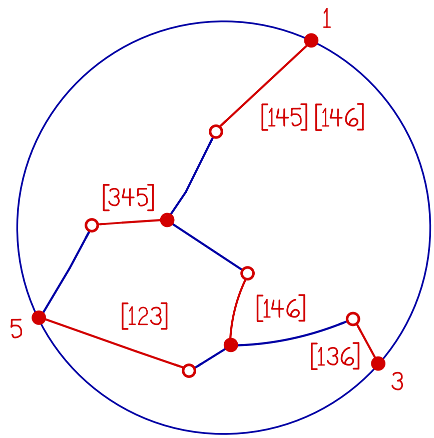


$\Gamma_{\mathscr{D}}$



$\Gamma_{\mathscr{D}}^{I}$ where $I = \{2,4,6\}$





$[145][146][136]\cdot[136][234][145]$ $\qquad$ $[145][146][136]\cdot[123][345][146]$

$$\Delta_{\mathscr{D}}^{246} = [145][146][136]\cdot\Big(\underbrace{[136][234][145]+[123][345][146]}\Big)$$

$= [134]\,[134]'$ exchange relation

$= [134]\,\overleftarrow{[246]}$ twist

<u>remark</u>: Δ^I is <u>independent</u> of blow-ups/downs, i.e. $\Delta^I = \tilde{\Delta}^I$ whenever Γ and $\tilde{\Gamma}$ are two bipartite graphs related by blow-ups/downs and both equipped with common <u>face-induced</u> <u>edge-weights</u>

<u>Theorem:</u> let $\mathscr{D}$ be any $\pi_{k,n}$-postnikov diagram and let $\Gamma_{\mathscr{D}}$ be its bipartite dual graph then

$$(*) \quad \overleftarrow{[I]} = \Delta^I_{\mathscr{D}} \cdot \underbrace{\prod_{\substack{\text{interior faces} \\ f \text{ in } \Gamma_{\mathscr{D}}}}}_{\text{denote by } \prod_{\mathscr{D}}} [I_f]^{-1} \quad \text{for any } k\text{-subset } I$$

<u>Comments on Proof:</u>

(1) prove $(*)$ for a specific $\pi_{k,n}$-diagram $\breve{\mathscr{D}}$

(involves condensation identities)

(2) prove that if $(*)$ is valid for some $\pi_{k,n}$-diagram $\mathscr{D}$ then $(*)$ is valid for any diagram obtained from $\mathscr{D}$ using a quad-move (spider-move)

Point 1:

- Construct $\pi_{k,n}$-diagram $\mathcal{D}$ such that

  (1) there is a k-subset $\breve{I}$ for each $[I] \in \mathbb{X}_{\mathcal{D}}$ such that $\overleftarrow{[I]} = m_I [\breve{I}]$ where m_I is some monomial of <u>frozen</u> plücker coordinates

  (2) $\mathbb{X}_{\breve{\mathcal{D}}} = \left\{ [\breve{I}] \;\middle|\; \begin{array}{l} \text{k-subset labels } I \text{ of} \\ \text{alter. regions of } \mathcal{D} \end{array} \right\}$ is the cluster of another $\pi_{k,n}$-diagram $\breve{\mathcal{D}}$

- $\Gamma^{I}_{\breve{\mathcal{D}}}$ has exactly one dimer configuration and formula (*) is valid for each $[I] \in \mathbb{X}_{\mathcal{D}}$, i.e.

$$\overleftarrow{[I]} = \Delta^{I}_{\breve{\mathcal{D}}} \prod\nolimits_{\breve{\mathcal{D}}}$$

- $\left\{ \Delta^{J}_{\breve{\mathcal{D}}} : \text{k-subsets } J \right\}$ satisfy the plücker relations

- $\left\{ \overleftarrow{[J]} : \text{k-subsets } J \right\}$ satisfy the plücker relations

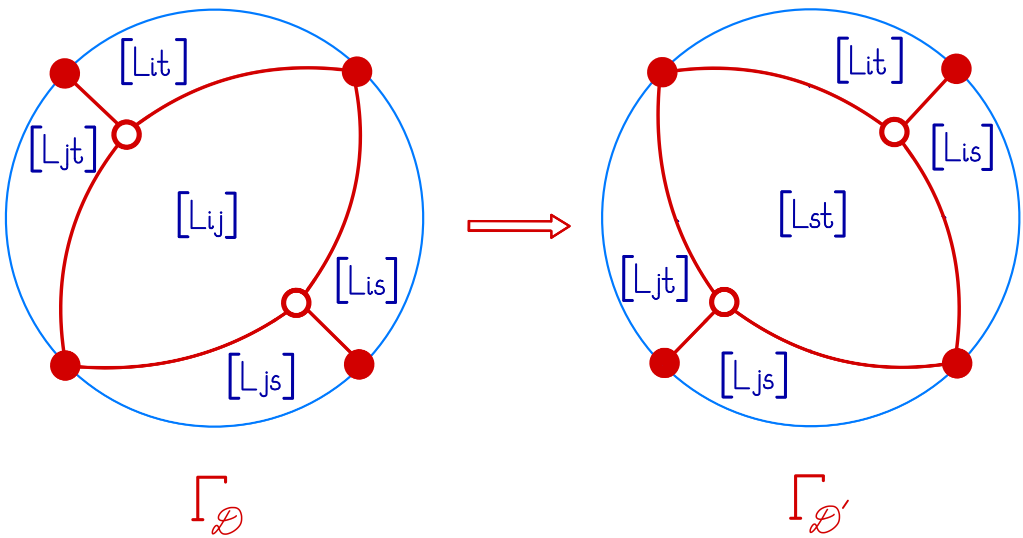
- each $\overleftarrow{[J]}$ is a homogeneous, degree one Laurent polynomial in $\overleftarrow{[I]}$ for $[I] \in \mathbb{X}_{\mathcal{D}}$

Point 2:

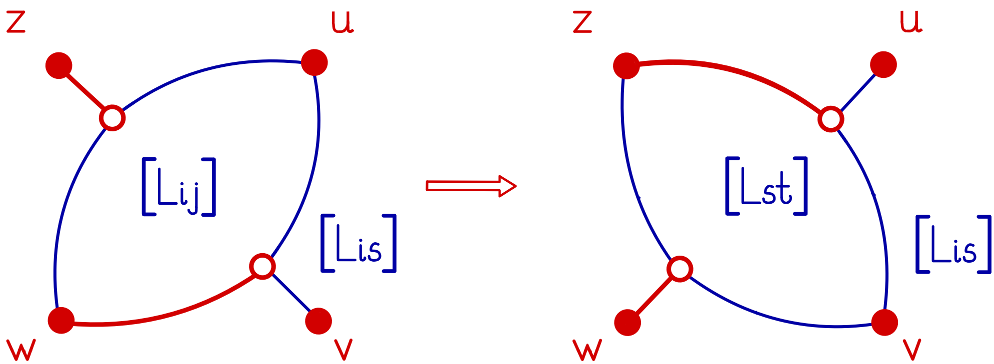
- must show $\Delta^{I}_{\mathscr{D}} \Pi_{\mathscr{D}} = \Delta^{I}_{\mathscr{D}'} \Pi_{\mathscr{D}'}$ whenever $\mathscr{D}$ and $\mathscr{D}'$ are related by a quad-move, i.e.

$$[L_{st}]\, \Delta^{I}_{\mathscr{D}} = [L_{ij}]\, \Delta^{I}_{\mathscr{D}'}$$

where $[L_{ij}] \in \mathcal{X}_{\mathscr{D}}$ and $[L_{st}] \in \mathcal{X}_{\mathscr{D}'}$ are exchanged by the quad-move with $i < s < j < t$ disjoint from L



- seven cases to consider: condition by how vertices u, v, w, z match with the two interior ○-vertices



$$\Gamma_{\mathscr{D}}^{I_{wz}} \qquad \Gamma_{\mathscr{D}'}^{I_{wz}}$$

$$[L_{ij}][L_{is}]\,\Delta_{\mathscr{D}}^{I_{wz}} \qquad [L_{st}][L_{is}]\,\Delta_{\mathscr{D}'}^{I_{wz}}$$

conditioned partition functions equal