Dimers and Grassmannians

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3rd Lecture

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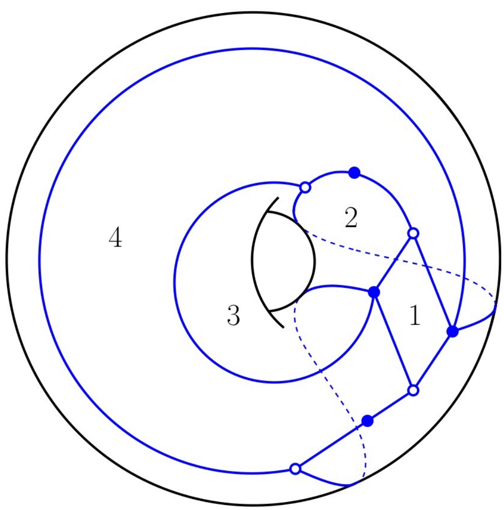
Dimers, surfaces, and spin-structures

definition: a bipartite map (Γ, Σ) is a pair where

- Γ is a bipartite graph, $\#\,\bullet$-vertices $= \#\,\circ$-vertices
- Σ is a closed, oriented surface, genus $= g$
- embedding $\Gamma \hookrightarrow \Sigma$, $\forall f$ face, closure $\bar{f} \simeq$ disk
- there exists abstract weight $\omega(e)$, $\forall e$ edge

Example: bipartite map (Γ, Σ) on $\Sigma =$ torus



definition: as before, the weight of a dimer configuration δ of Γ is $\omega(\delta) = \prod_{e \in \delta} \omega(e)$

observation: the symmetric difference $\delta \triangle \delta'$ of two dimers δ and δ' is disjoint union of simple cycles in Σ, and determines homology class

$$[\delta \triangle \delta'] \in H_1(\Sigma; \mathbb{F}_2)$$

definition: for $\gamma \in H_1(\Sigma; \mathbb{F}_2)$ the γ-dimer partition function is

$$\Delta_\Gamma^\gamma = \sum_{\substack{\text{dimers } \delta \text{ on } \Gamma \\ [\delta \triangle \delta_0] = \gamma}} \omega(\delta)$$

where δ_0 is a fixed reference dimer

<u>definition:</u> a <u>kasteleyn orientation</u> $\mathcal{K}$ of (Γ, Σ) is an edge-orientation $\mathcal{K}(e) \in \{\pm 1\}$ $\forall e$ edges such that

$$\prod_{e \in \partial f} \mathcal{K}(e) = (-1)^{1 + \frac{|\partial f|}{2}} \qquad \forall f \text{ faces}$$

where $\mathcal{K}(e) = \begin{cases} 1 & \bullet\!\leftarrow\!\circ \\ -1 & \bullet\!\rightarrow\!\circ \end{cases}$

<u>remark:</u> if $\mathcal{K}$ is kasteleyn and $v \in \Gamma$ is a vertex then

$$\tilde{\mathcal{K}}(e) = \begin{cases} -\mathcal{K}(e) & \text{if } v \in \partial e \\ \mathcal{K}(e) & \text{if } v \notin \partial e \end{cases}$$

defines a kasteleyn orientation; $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are said to be <u>gauge-equivalent</u>

<u>remark:</u> label edges $e \in \delta_0$ by $\{1, \ldots, n\}$ and label a vertex $v \in \Gamma$ by $i \in \{1, \ldots, n\}$ if $v \in \partial e$ and e is labeled by i where $\# \Gamma = 2n$

definition: the kasteleyn matrix $A_\Gamma^{\mathcal{K}}$ is the $n \times n$ matrix whose $i \times j$ entry is

$$\mathcal{K}\big(\underset{i}{\circ}\!-\!\!-\!\underset{j}{\bullet}\big) \cdot \omega\big(\underset{i}{\circ}\!-\!\!-\!\underset{j}{\bullet}\big)$$

Theorem: (Cimasoni-Reshetikhin)

$$\det A_\Gamma^{\mathcal{K}} = \mathcal{K}(\delta_0) \cdot \sum_{\gamma \in H_1(\Sigma; \mathbb{F}_2)} (-1)^{q^{\mathcal{K}}(\gamma)} \Delta_\Gamma^{\gamma}$$

where $\mathcal{K}(\delta_0) = \prod_{e \in \delta_0} \mathcal{K}(e)$ and $q^{\mathcal{K}} : H_1(\Sigma; \mathbb{F}_2) \to \mathbb{F}_2$

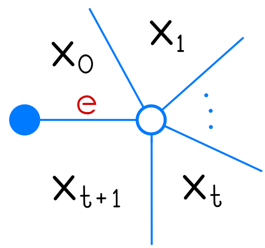
is an $\mathbb{F}_2$-valued quadratic form depending on both the reference dimer δ_0 and kasteleyn orient. $\mathcal{K}$

remark: Kuperberg has a construction, using δ_0 and $\mathcal{K}$, which creates a vector field η on Σ with even-index singularities. From η a cohomology class $\xi \in H^1(P_{SO(2)}; \mathbb{F}_2)$ where $P_{SO(2)} \to \Sigma$ is a principal $SO(2)$-bundle, i.e. a *spin-structure*

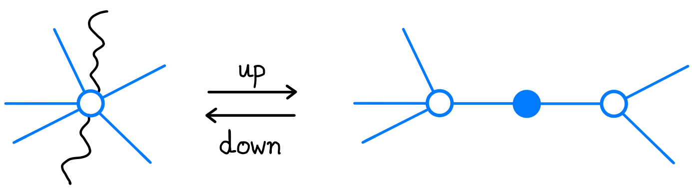
Mutation

definition: let $\mathbb{X}_\Gamma = \{x_f : \text{faces of } (\Gamma, \Sigma)\}$ be a set of independent variables. Let $\omega(e)$ be the associated face-induced edge-weight for an edge e of (Γ, Σ)

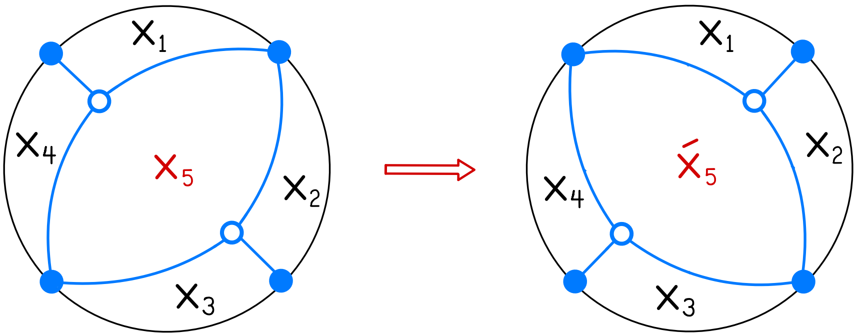
$$\omega(e) = \prod_{s=1}^{t} x_s \quad \text{where}$$



remark: we work up to isotopy and ○-vertex blow-ups and blow-downs



__definition:__ let (Γ, Σ) be a bipartite map with face-induced edge-weights. the spider-move associated to a quadrilateral face is local transformation



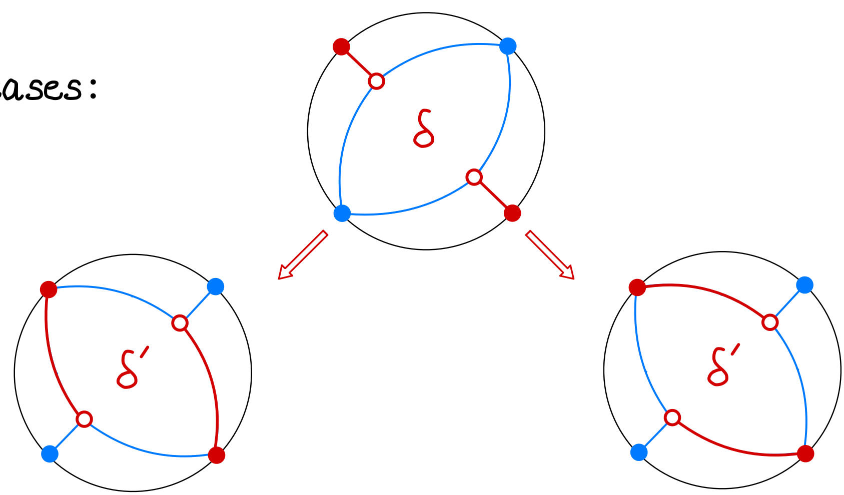
(Γ, Σ) $(\acute{\Gamma}, \Sigma)$

$$x_5 \acute{x}_5 = x_1 x_3 + x_2 x_4$$

exchange relation

__definition:__ using the spider-move we can transport a dimer δ on (Γ, Σ) to a dimer δ' on $(\acute{\Gamma}, \Sigma)$

seven cases:

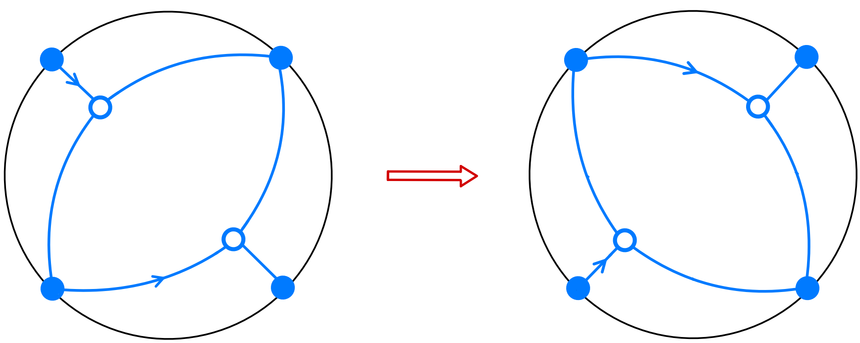


<u>observation:</u> let δ_0 be a reference dimer for (Γ, Σ) and let δ_0' be a dimer for $(\acute{\Gamma}, \Sigma)$ obtained by transporting δ_0 using the spider-move then

$$\Pi_\Gamma \Delta_\Gamma^\gamma = \Pi_{\acute{\Gamma}} \Delta_{\acute{\Gamma}}^\gamma \quad \text{for any } \gamma \in H_1(\Sigma; \mathbb{F}_2)$$

where $\Pi_\Gamma = \prod_{\text{faces } f} x_f^{-1}$

<u>definition:</u> we can also transport a kasteleyn orient. $\mathcal{K}$ on (Γ, Σ) to a kasteleyn orient. $\acute{\mathcal{K}}$ on $(\acute{\Gamma}, \Sigma)$



(Γ, Σ) with $\mathcal{K}$ $\qquad\qquad$ $(\acute{\Gamma}, \Sigma)$ with $\acute{\mathcal{K}}$

<u>observation:</u> the spider-move on kasteleyn orients. is well-defined and is an <u>involution</u>

Conjecture: let δ_0 and $\mathcal{K}$ be a reference dimer and kasteleyn orient. on (Γ, Σ) ; let δ_0' and $\mathcal{K}'$ be their counterparts on (Γ', Σ) then the quadratic forms $q^{\mathcal{K}}$ and $q^{\mathcal{K}'}$ are equal