Categorification TA / Problem Solving Session

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based on lectures by

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Dimers Summer School

CUNY

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Birthdays!

Go to the table that corresponds to the day of your birthday mod 12

Introduce yourself :)

Plan for the session:

- brief overview of lecture
- group discussion of notes
- exercises in groups

Goal: meet people & practice with concepts to better learn the material :)

Summary of Categorification Lecture 1:

The main goal of lecture 1 was to prime you for the definition of the

Cluster category

This is a technical definition more easily understood in layers:

Layer 1: quivers

$$\begin{tikzcd} & & 3 \arrow[dl] \\ 1 \arrow[r] & 2 \arrow[dr] & \\ & & 4 \end{tikzcd}$$

quiver representations	cluster algebras
linear algebra	Combinatorics

Quiver Representations:

A quiver representation is an assignment of

vector spaces — vertices

linear maps — arrows

"indecomposables"

Goal: understand building blocks of quiver representations

We use categorification:

quiver reps $\cong$ modules over the path algebra

are the same as

Think: first approximation of categorification of cluster algebras

Layer 2: Quiver of Quiver Representations:

We create the Auslander-Reiten quiver to study all indecomposable quiver repns.

Layer 2: indecomposable quiver repns

$$\begin{tikzcd}
 & & \Bbbk^{3} \arrow[dl, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] \\
\underset{1}{\Bbbk} \arrow[r, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & \underset{2}{\Bbbk^2} & \\
 & & \Bbbk^{4} \arrow[ul, "\begin{bmatrix} 1 \\ 1 \end{bmatrix}"']
\end{tikzcd}$$

AR Quiver

vertices: indecomps.
arrows: irreducible morphisms

Layer 3: Cluster Category:

We connect the world of quiver representations to cluster algebras with the cluster category!

quivers

Q

quiver repns

$\operatorname{rep} Q \cong \operatorname{mod} \Bbbk Q/I$

Cluster algebras

$\mathcal{A}(Q)$

AR quiver

cluster category

$\mathcal{C}_Q = \mathcal{D}^b(\operatorname{rep} Q)/\tau^{-1}[1]$

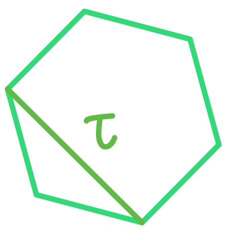
categorification

more on this later in lecture series

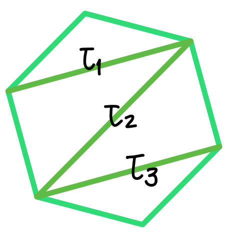
Another Way to Think About Quivers, Cluster Algebras & Representations

A **geometric model** in type A_n makes visualizing these structures really nice!

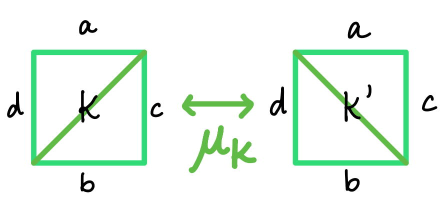
Polygon Model: we can see cluster algebras.



arc τ — cluster variable x_τ



triangulations $T = \{\tau_1, \tau_2, \tau_3\}$ — clusters $\{x_{\tau_1}, x_{\tau_2}, x_{\tau_3}\}$



Ptolemy relation:
$kk' = ab + cd$

— mutation

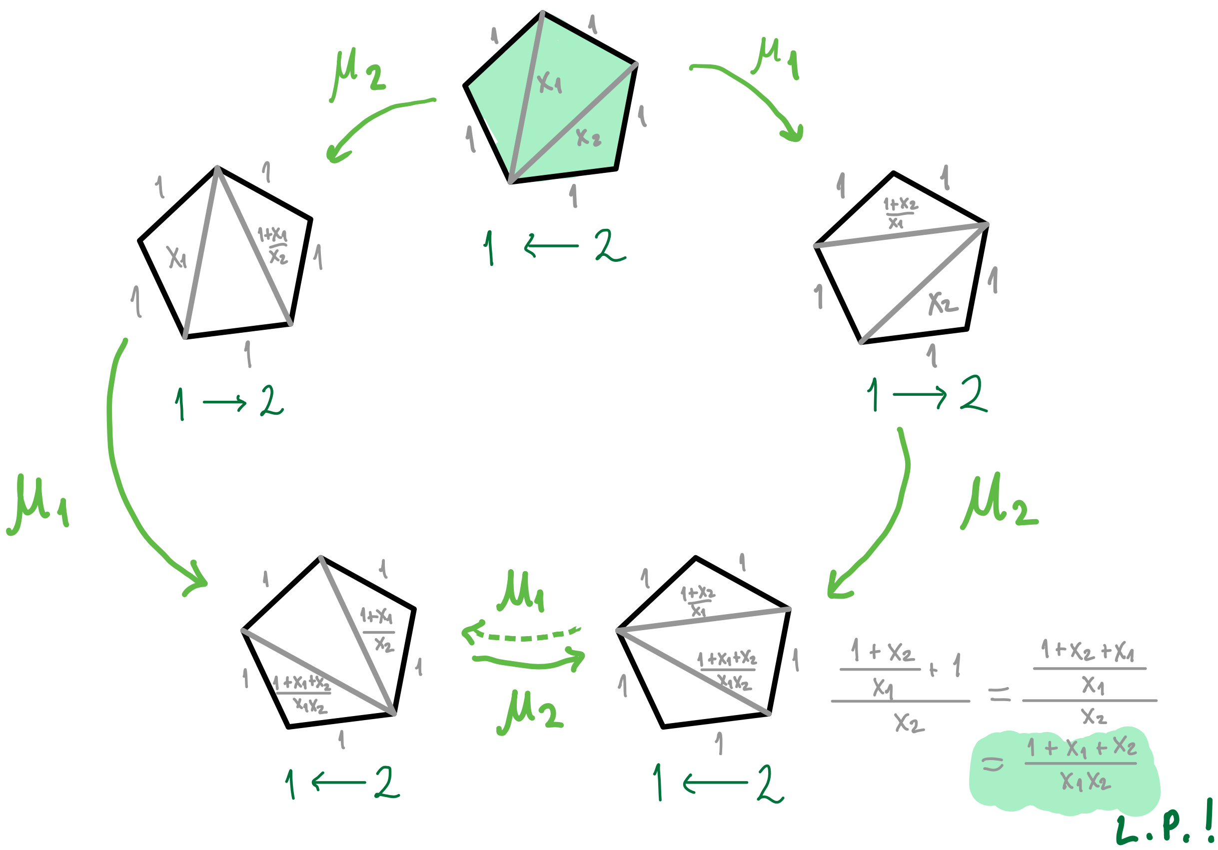
$$\{x_{\tau_1}, \dots, x_{\tau_k}, \dots, x_{\tau_n}\} \xrightarrow{\mu_k} \{x_{\tau_1}, \dots, x_{\tau_k'}, \dots, x_{\tau_n}\}$$

Example of A_2 Cluster Algebra:

We can start with the highlighted pentagon & weight the boundary arcs 1.

Then apply Ptolemy relation to express other arcs in terms of x_1, x_2:

Cluster Exchange Graph for A_2



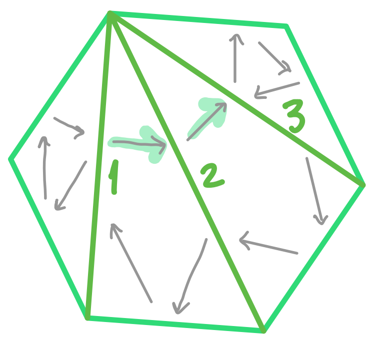
Cluster variables: $\left\{x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\right\}$

Type A Quiver Representations from Polygons:

We associate a quiver to a triangulation:

vertices: diagonals in T
arrows: Clockwise orientation in Δs in T

Ex

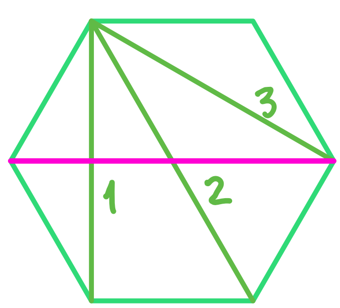


$\rightsquigarrow \; Q_T = 1 \to 2 \to 3$

We associate a quiver repn to a diagonal γ not in T by

- put a 1-dimensional vector space on every vertex that corresponds to a diagonal in T crossed by γ

Ex



$\rightsquigarrow \; M(\gamma) = \mathbb{k} \xrightarrow{1} \mathbb{k} \xrightarrow{0} 0$

Remarks about Quiver Representations from Polygons:

The quiver repns we obtain are indecomposable & completely determined by their dimension vector ($\underline{\dim} = (\dim(M_i))_{i \in Q_0}$).

Caveat: if our initial triangulation T produces an oriented 3-cycle in Q_T, we need to use quivers with potential (also known as bound quivers)

$\hookrightarrow$ we will see this tomorrow!

AR Quiver for A_3:

Let Q be the A_3 quiver $Q = 1 \longrightarrow 2 \longrightarrow 3$.
There are exactly six isomorphism classes of indecomposable representations:

$$\mathbb{k} \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathbb{k} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{k}$$

$S(1) = I(1)$ $\qquad$ $S(2)$ $\qquad$ $S(3) = P(3)$

$$\mathbb{k} \xrightarrow{1} \mathbb{k} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathbb{k} \xrightarrow{1} \mathbb{k} \qquad \mathbb{k} \xrightarrow{1} \mathbb{k} \xrightarrow{1} \mathbb{k}$$

$I(2)$ $\qquad$ $P(2)$ $\qquad$ $I(3) = P(1)$

Using our symbolic notation, we have:

$S(1) = I(1) = \mathbf{1}$; $S(2) = \mathbf{2}$; $S(3) = P(3) = \mathbf{3}$;

$I(2) = \begin{matrix}1\\2\end{matrix}$; $P(2) = \begin{matrix}2\\3\end{matrix}$; $I(3) = P(1) = \begin{matrix}1\\2\\3\end{matrix}$

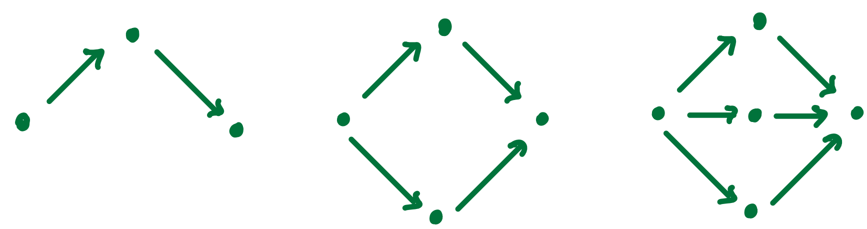
These form the vertices of the AR quiver.
To get the arrows, we need to compute irreducible morphisms.

Think: morphisms that do NOT factor through another representation

AR Quiver for A_3 Cont:

Khrystyna showed us the Auslander-Reiten translation τ that we can use to compute the AR quiver.

In type A_n, we can use the knitting algorithm that is a quick trick to computing meshes in the AR quiver.



① Start with indecomposable projectives:

$$P(1) = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \; ; \; P(2) = \begin{smallmatrix}2\\3\end{smallmatrix} \; ; \; P(3) = 3$$

② Draw $P(i) \longrightarrow P(j)$ when $j \to i$ in Q_1 so that each projective sits at a different level:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] & \\
3 \arrow[ur] & &
\end{tikzcd}$$

AR Quiver for A_3 Cont:

③ (Knitting) Complete meshes based on dimension. In type A_n, we'll have:

$$\begin{tikzcd} & M_1 \\ L \arrow[ur] \arrow[dr] & \\ & M_2 \end{tikzcd} \rightsquigarrow \begin{tikzcd} & M_1 \arrow[dr] & \\ L \arrow[ur] \arrow[dr] & & \tau^{-1}L \\ & M_2 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} & M_1 \\ L \arrow[ur] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} & M_1 \arrow[dr] & \\ L \arrow[ur] & & \tau^{-1}L \end{tikzcd}$$

$$\begin{tikzcd} L \arrow[dr] & \\ & M_2 \end{tikzcd} \rightsquigarrow \begin{tikzcd} L \arrow[dr] & & \tau^{-1}L \\ & M_2 \arrow[ur] & \end{tikzcd}$$

We require $\underbrace{\underline{\dim} L + \underline{\dim}\, \tau^{-1}L}_{\text{horizontal dimn}} = \underbrace{\underline{\dim} M_1 + \underline{\dim} M_2}_{\text{vertical dimn}}$

horizontal dimn = vertical dimn

$$\begin{tikzcd} & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & \\ & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ur] & & 1 \end{tikzcd}$$

c.f. quiver repns by Ralf Schiffler ♡