"Dimers & Grassmannians" Problem / Example Session 1

① Stiefel-Plücker correspondence & injectivity of Plücker embedding.

$$\begin{array}{cc} & m \\ k & \boxed{\quad A \quad} \end{array} \iff \begin{array}{cc} & k+m \\ k & \left[\begin{array}{cccc|c} & & & 1 & \\ & & -1 & & \\ & 1 & & & A \\ \pm 1 & & & & \end{array}\right] \end{array} \leftarrow \tilde{A}$$

$\Delta_{X,Y}$ denotes minor of A w/ row set $X \subseteq [k]$, col set $Y \subseteq [m]$

$$I = (w_0(X))^{comp} \sqcup (Y+k)$$

$w_0(z) \to k+z-1$, comp = complement in $\{1,\dots,k\}$

$Y+k = \{y+k \mid y \in Y\}$

Claim/Observ $\underset{\text{from } A}{\Delta_{X,Y}} = [I]_{\tilde{A}}$

Ex $A = \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}$

$\Delta_{12,23} = \begin{vmatrix} 2 & 5 \\ -1 & 3 \end{vmatrix} = 11$

$w_0(1) = 3 \quad w_0(2) = 2$

$w_0(12) = 23 \quad (23)^c = 1$

$\tilde{A} = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 2 & 5 \\ 0 & -1 & 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 2 & 1 & -2 \end{array}\right)$

$23 + 3 \longrightarrow 56$

$[156]_{\tilde{A}} = \begin{vmatrix} 0 & 2 & 5 \\ 0 & -1 & 3 \\ 1 & 1 & -2 \end{vmatrix} = \Delta_{12,23}$ ✓

$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Recall Plücker embedding

set of k-subsets of $\{1, \dots, n\}$

$$GL_k \backslash \mathrm{Mat}^*_{k,n} \longrightarrow \mathbb{CP}^{\binom{n}{k}-1}$$

$$\begin{array}{c} A \\ \cap \\ \mathrm{Mat}^*_{k,n} \end{array} \longrightarrow \left[[I]_A \;\middle|\; I \in \binom{[n]}{k} \right]$$

$\mathcal{S}$ A & $B \in \mathrm{Mat}^*_{k,n}$ have the same image.

We can assume $[1..k]A = [1..k]B \neq 0$

So we can find $g_A, g_B \in GL_k$ s.t.

$$\tilde{A} = g_A \cdot A = \left(\begin{smallmatrix} & & 1 \\ & \iddots & \\ \pm 1 & & \end{smallmatrix} \middle| X \right)$$

$$\tilde{B} = g_B \cdot B = \left(\begin{smallmatrix} & & 1 \\ & \iddots & \\ \pm 1 & & \end{smallmatrix} \middle| Y \right)$$

$$\tilde{A} = g_A \cdot A = \left(\begin{smallmatrix} & & -1 \\ & \cdot^{\cdot^{\cdot}} & \\ \pm 1 & & \end{smallmatrix} \middle| X \right)$$

$$\tilde{B} = g_B \cdot B = \left(\begin{smallmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ \pm 1 & & & \end{smallmatrix} \middle| Y \right)$$

By assumption,

$$[I]_{\tilde{A}} = [I]_{\tilde{B}} \quad \forall I \in \binom{[n]}{n}$$

$$I \xrightarrow[\text{S-P}]{} R_I, C_I \longrightarrow \Delta_{R_I, C_I}(X) = \Delta_{R_I, C_I}(Y)$$

$n = 8$, $k = 4$, $I = \{2, 5, 6, 8\}$

comp $\downarrow$ 1, 3, 4 $\downarrow$ 1, 2, 4

shift back by 4 $\downarrow$ 1, 2, 4

$$[2568]_{\tilde{A}} = [2568]_{\tilde{B}} \iff \Delta_{124,124}(X) = \Delta_{124,124}(Y)$$

$\boxed{\text{By varying } I, \text{ conclude } X = Y.}$

So we know $x = y$ $\qquad \tilde{A} = \left(\cdots \middle| x \right) \quad \tilde{B} = \left(\cdots \middle| y \right)$

$$\tilde{A} = g_A \cdot A \qquad \tilde{B} = g_B \cdot B \qquad\qquad \tilde{B} = \left(\cdots \middle| x \right)$$

so $A = g_A^{-1} g_B B$

Recall Short Plücker: $L \in \binom{[n]}{k-2}$ $\quad i < j < l < t$

$$[Lil][Ljt] = [Lij][Llt] + [Lit][Ljl]$$

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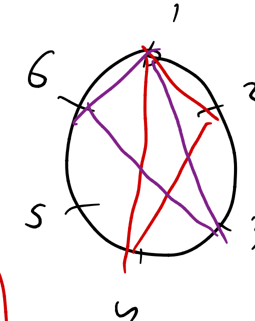
$$[Lil][Ljt] = [Lij][Llt] + [Lit][Ljl]$$



$$\left(\begin{array}{ccc|ccc} & & 1 & x_{11} & x_{12} & x_{13} \\ & -1 & & x_{21} & x_{22} & x_{23} \\ 1 & & & x_{31} & x_{32} & x_{33} \end{array}\right)$$

$$\underbrace{\qquad\qquad\qquad}_{X}$$

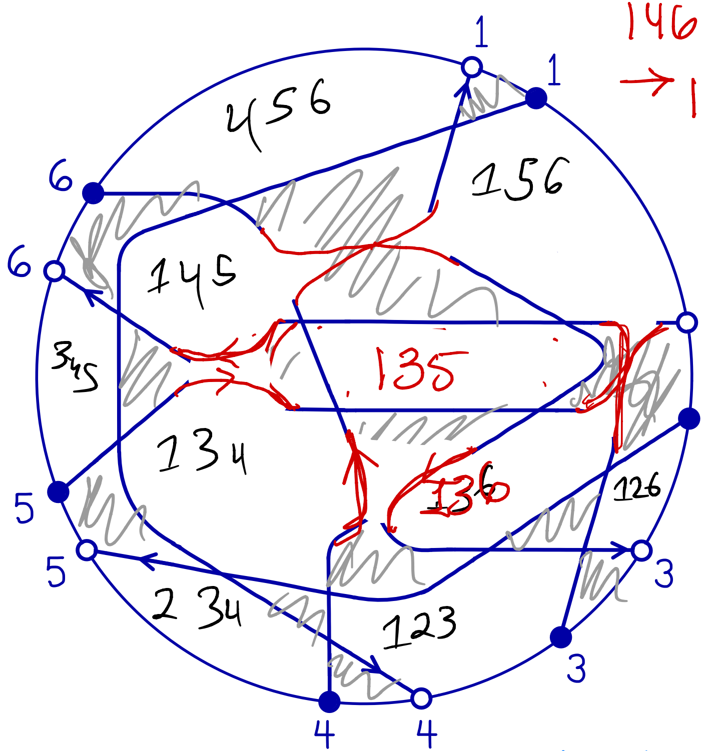
$$\begin{aligned}
\Delta_{1,1}(x) = x_{11} &\overset{SP}{=\!=} [124] \\
x_{13} &= [126] \\
x_{21} &= [134] \\
x_{23} &= [136]
\end{aligned}$$

$$\Delta_{12,13} \longrightarrow [146]$$

$$1 \longrightarrow [123]$$

$$\Delta_{12,13} \cdot 1 = x_{11}x_{23} - x_{21}x_{13} \rightsquigarrow [124][136] = [146][123] + [134][126]$$

"L" $= \{1\}$



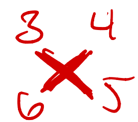
146 → 135

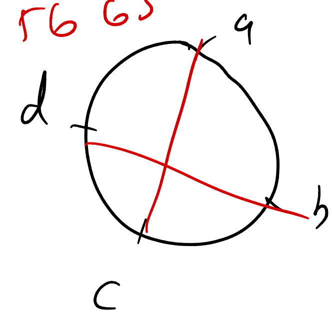
$L = \{1\}$

$I, J \in \binom{[n]}{k}$ have a crossing

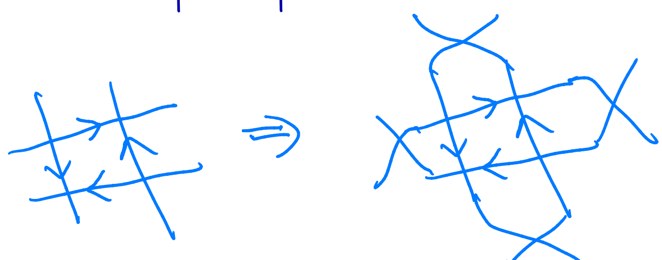
if $\exists$ $a, b \in I \setminus J$
$c, d \in J \setminus I$

34 45 56 63

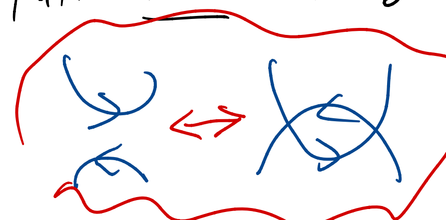


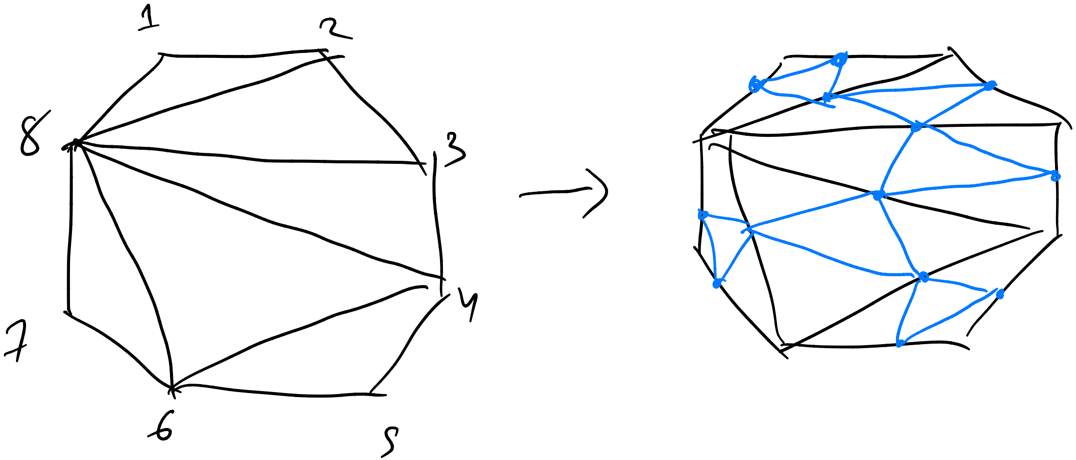


Labeling Rule:
all alternating regions to
Left of path from i get
label i.



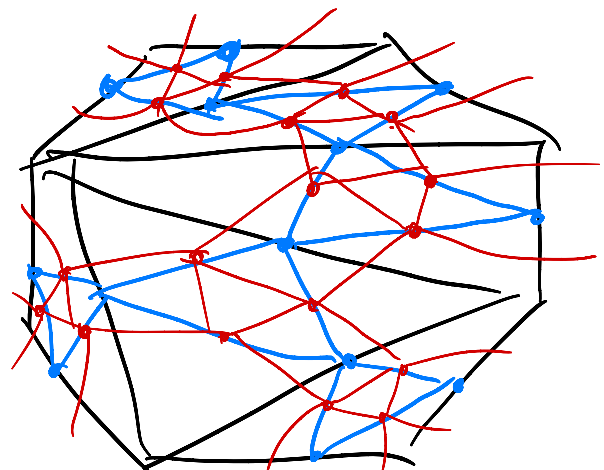
Equiv



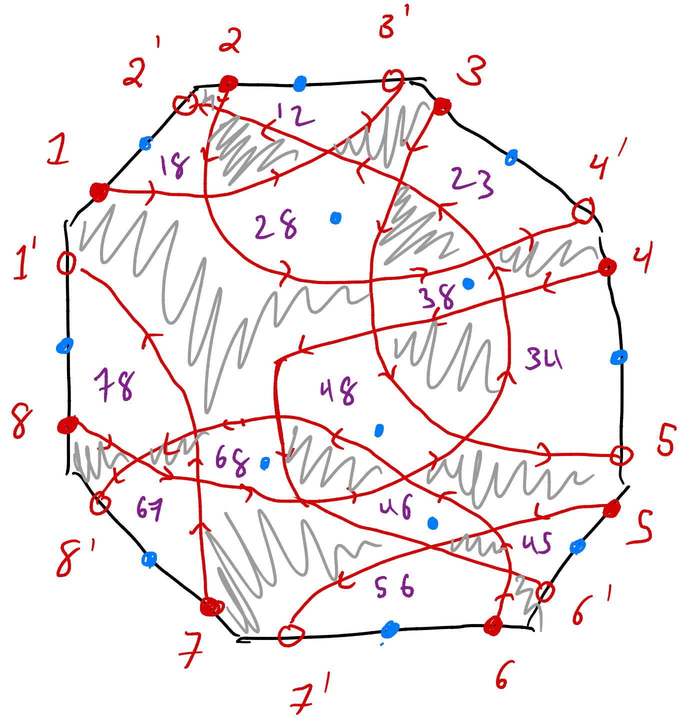


{28, 38, 48, 46, 68}

12, 23, ...



Better Picture on next Page.



1
1'
2
2'
3
3'
4
4'
5
5'
6
6'
7
7'
8
8'
12
18
23
28
38
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78
68
67
46
45
56