Categorification

Dimers Summer School

CUNY

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Overview

* Quiver Representations
* Cluster Algebras
* Categorification

Path Algebras

* $Q = (Q_0, Q_1)$ quiver - directed graph
  Q_0 - set of vertices
  Q_1 - set of arrows

* $\Bbbk$ - algebraically closed field

* $\Bbbk Q$ - path algebra: associative algebra over $\Bbbk$
  with basis - set of all paths in Q
  multiplication - composition of paths

ex $Q : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

$\Bbbk Q$ has basis $\{\alpha, \beta, \alpha\beta, \overbrace{e_1, e_2, e_3}^{\text{constant paths at each vertex}}\}$

$\Bbbk Q = \{\lambda_1\alpha + \lambda_2\beta + \lambda_3\alpha\beta + \lambda_4 e_1 + \lambda_5 e_2 + \lambda_6 e_3 \mid \lambda_i \in \Bbbk\}$

multiplication: $\alpha \cdot \beta = \alpha\beta$, $\alpha \cdot \alpha = 0$, $e_1 \cdot \alpha = \alpha$, $e_i^2 = e_i$

Path Algebras

* $\Bbbk Q$ - path algebra
* $I \subset \Bbbk Q$ - two sided ideal
* $\boxed{A = \Bbbk Q / I}$ - path algebra of a quiver with relations or bound quiver algebra

ex $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ $\qquad I = \langle \alpha\beta \rangle$

$A = \Bbbk Q / I$ has basis $\{\alpha, \beta, e_1, e_2, e_3\}$

multiplication: $\alpha \cdot \beta = 0$

Path Algebras

$\left\{\begin{array}{l}\text{finite dimensional} \\ \text{basic associative} \\ \text{algebra } A \text{ over } \Bbbk\end{array}\right\} \overset{\text{bij}}{\longleftrightarrow} \left\{\begin{array}{l}\text{path algebras} \\ \text{with relations} \\ \Bbbk Q/I \text{ where} \\ I \text{ is admissible}\end{array}\right\}$

$\text{mod}\, A \cong \text{rep}(Q, I)$

category of finite dimensional A-modules

category of quiver representations

* if A is not basic then $A \rightsquigarrow A^{\text{basic}}$ s.t. $\text{mod}\, A \cong \text{mod}\, A^{\text{basic}}$

* I is admissible if $(\text{arrow ideal})^m \subset I \subset (\text{arrow ideal})^2$

Quiver Representations

* rep$(Q, I) \cong$ mod A

objects : quiver representations

$$M = (M_i, \varphi_\alpha)_{\substack{i \in Q_0 \\ \alpha \in Q_1}}$$

M_i - $\mathbb{k}$ vector space
φ_α - linear map b/w them
s.t. φ's satisfy relations of I

ex $\mathbb{k}(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3) / \langle \alpha\beta \rangle$

$$M = \mathbb{k}^{a_1} \xrightarrow{\varphi_\alpha} \mathbb{k}^{a_2} \xrightarrow{\varphi_\beta} \mathbb{k}^{a_3}$$

$\varphi_\alpha \in \mathrm{Mat}_{a_2 \times a_1}(\mathbb{k})$

$\varphi_\beta \in \mathrm{Mat}_{a_3 \times a_2}(\mathbb{k})$

$\varphi_\beta \varphi_\alpha = 0$

Quiver Representations

* $\mathrm{rep}(Q, I) \cong \mathrm{mod}\,A$

morphisms: $f : M \to M'$ is a collection of linear maps $f = (f_i)_{i \in Q_0}$ s.t. for every arrow $\alpha \in Q_1$

$$\begin{tikzcd} M: & M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowright"] & M_j \arrow[d, "f_j"] \\ M': & M'_i \arrow[r, "\varphi'_\alpha"'] & M'_j \end{tikzcd}$$

the square commutes

- f is an isomorphism if all f_is are isomorphisms

Quiver Representations

* **direct sum:** $M \oplus M' := \left(M_i \oplus M'_i \, , \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} \right)_{\substack{i \in Q_0 \\ \alpha \in Q_1}}$

* M is **indecomposable** if whenever $M \cong M' \oplus M''$ then $M' = 0$ or $M'' = 0$, and $M \neq 0$

* mod$A$ is <u>Krull-Schmidt</u>:

  for any $M \in \text{mod}A$ we can write
  $$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t$$
  where each M_i is indecomposable and unique up to reordering

Quiver Representations

Exercise: $A = \Bbbk Q / I$

what is the action of A on the quiver representation $M \in \mathrm{rep}(Q, I)$?

Quiver Representations

ex $A = \Bbbk\left(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3\right) / \langle \alpha\beta \rangle$

$M \in \operatorname{mod} A$: $\quad M = \Bbbk^{a_1} \xrightarrow{\varphi_\alpha} \Bbbk^{a_2} \xrightarrow{\varphi_\beta} \Bbbk^{a_3}$ $\qquad \varphi_\beta \varphi_\alpha = 0$

ind A:

$$\begin{array}{lcl}
\Bbbk \to 0 \to 0 & & 1 \\
0 \to \Bbbk \to 0 & \xrightarrow{\text{notation}} & 2 \\
0 \to 0 \to \Bbbk & & 3 \\
\Bbbk \xrightarrow{1} \Bbbk \to 0 & & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \\
0 \to \Bbbk \xrightarrow{1} \Bbbk & & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}
\end{array}$$

$\left.\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\}$ simple modules

Quiver Representations

* an Auslander-Reiten (AR)-quiver of mod A has vertices ⇝ ind. A-modules
  arrows ⇝ irreducible morphisms

$$\begin{tikzcd} & * \arrow[dr] & \\ \tau M \arrow[ur] \arrow[r] \arrow[dr] & * \arrow[r] & M \\ & * \arrow[ur] & \end{tikzcd}$$

τ not projective

ex $A = \Bbbk(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3) / \langle \alpha\beta \rangle$

$$\begin{tikzcd}[column sep=small] & \boxed{0 \to \Bbbk \xrightarrow{1} \Bbbk} \arrow[dr] & & & \\ \boxed{0 \to 0 \to \Bbbk} \arrow[ur] & & \boxed{0 \to \Bbbk \to 0} \arrow[ll, dashed, "\tau"'] \arrow[dr] & & \boxed{\Bbbk \to 0 \to 0} \arrow[ll, dashed, "\tau"'] \\ & & & \boxed{\Bbbk \xrightarrow{1} \Bbbk \to 0} \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[dr] & & & \\ 3 \arrow[ur] & & 2 \arrow[ll, dashed, "\tau"] \arrow[dr] & & 1 \arrow[ll, dashed, "\tau"'] \\ & & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] & \end{tikzcd}$$

Quiver Representations

* $A_A = \bigoplus_{i \in Q_0} P(i)$ $P(i)$ indecomposable projective at vertex i

* $P(i) = (P(i)_j, \varphi_\alpha)_{\substack{j \in Q_0 \\ \alpha \in Q_1}}$

where $P(i)_j$ has basis of all paths starting at i and ending at j

φ_α - composition of a path with arrow α

ex $A = \Bbbk(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)/\langle \alpha\beta \rangle$

$P(1)$ has basis e_1, α $\rightsquigarrow$ $\Bbbk \xrightarrow{1} \Bbbk \to 0 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$

$P(2) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ $P(3) = 3$

Quiver Representations

* $D({}_A A) = \bigoplus_{i \in Q_0} I(i)$ $\quad$ $I(i)$ indecomposable injective at vertex i

* $I(i) = (I(i)_j, \varphi_\alpha)_{\alpha \in Q_1}^{j \in Q_0}$

  where $I(i)_j$ has basis of paths starting at j and ending at i

  φ_α - deletion of an arrow α from a path

ex $\quad A = \Bbbk(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)/\langle \alpha\beta \rangle$

$I(1)$ has basis e_1 $\rightsquigarrow$ $\Bbbk \to 0 \to 0$

$I(2)$ has basis α, e_2 $\rightsquigarrow$ $\Bbbk \to \Bbbk \to 0$

Quiver Representations

* $M \in \operatorname{mod} A$

$$\bigoplus_{j\in I_1} P(j) \xrightarrow{f} \bigoplus_{i\in I_0} P(i) \to M \to 0$$

projective presentation of M

$\wr$ Nakayama functor

$$\ker f^* \hookrightarrow \bigoplus_{j\in I_1} I(j) \xrightarrow{f^*} \bigoplus_{i\in I_0} I(i)$$

$$\ker f^* = \tau M$$

ex

$$A = \Bbbk(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)/\langle \alpha\beta \rangle$$

$$\text{Proj} = {\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}} \oplus {\scriptstyle\begin{smallmatrix}2\\3\end{smallmatrix}} \oplus 3$$

$\wr$ Nakayama

$$\text{Inj} = 1 \oplus {\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}} \oplus {\scriptstyle\begin{smallmatrix}2\\3\end{smallmatrix}}$$

$\tau(1)$?

$$\begin{tikzcd}
{\begin{smallmatrix}2\\3\end{smallmatrix}} \arrow[rr] \arrow[dr, two heads] \arrow[d, phantom, "\wr"] & & {\begin{smallmatrix}1\\2\end{smallmatrix}} \arrow[r] \arrow[d, phantom, "\wr"] & 1 \arrow[r] & 0 \\
{\begin{smallmatrix}1\\2\end{smallmatrix}} \arrow[rr] & 2 \arrow[ur, hook] & 1 & &
\end{tikzcd}$$

$\tau(1) = 2 \hookrightarrow {\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}} \to 1$

Quiver Representations

Exercise: Let $A_3 = \Bbbk(1 \to 2 \to 3)$

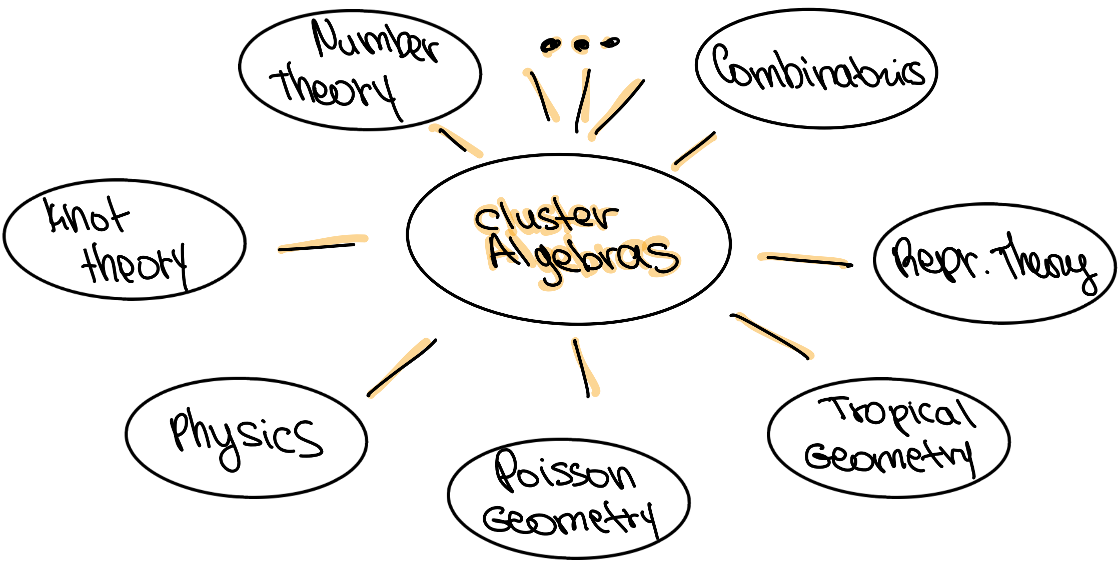
1) what are all indecomposable representations of A_3?

2) compute the AR-quiver of $\operatorname{mod} A_3$.

3)* generalize above to the case $A_n = \Bbbk(1 \to 2 \to 3 \to \cdots \to n)$

Cluster Algebras

* introduced by Fomin-Zelevinsky in 2001
* class of commutative rings in $\mathbb{Q}(x_1, \ldots, x_n)$



Cluster Algebras

* Q - quiver without 2-cycles and loops with vertices $1, \dots, n$
* **cluster algebra** $\mathcal{A}_Q \subset \mathbb{Q}(x_1, \dots, x_n)$ defined by a set of generators, called **cluster variables**, and satisfying **exchange relations** that are defined recursively
* **initial seed**: $(\underbrace{\mathbf{x}}_{\text{cluster}} = (x_1, \dots, x_n), Q)$
* **mutation** μ_k, $k \in [n]$, of a seed $(\mathbf{x}, Q)$ is a new seed $\mu_k(\mathbf{x}, Q) = (\mu_k(\mathbf{x}), \mu_k(Q))$
* $\mathcal{A}_Q$ is generated by all cluster variables obtained from the initial seed by sequences of mutations

Cluster Algebras

$$\boxed{\mu_k(\mathbb{X}, Q)} = (\mu_k(\mathbb{X}), \mu_k(Q))$$

let $\mathbb{X} = (f_1, \ldots, f_n)$

$$\mu_k(\mathbb{X}) = (f_1, \ldots, f_k', \ldots, f_n)$$

$$f_k' = \frac{\prod_{i \to k \in Q} f_i + \prod_{k \to i \in Q} f_i}{f_k}$$

$\mu_k(Q)$ is obtained from Q as follows:

1) $i \to k \to j$ add $i \to j$
2) reverse all arrows at k
3) remove 2-cycles

Cluster Algebras

ex $Q:$
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

$\mathcal{A}_Q \subset \mathbb{Q}(x_1, x_2, x_3)$

initial seed: $\left((x_1, x_2, x_3), \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd} \right)$

$$\Big\downarrow \mu_1$$

$$\left(\left(\frac{x_2+x_3}{x_1}, x_2, x_3\right), \begin{tikzcd} & 2 \arrow[dl] & \\ 1 \arrow[rr] & & 3 \end{tikzcd} \right)$$

$$\Big\downarrow \mu_2$$

$$\left(\left(\frac{x_2+x_3}{x_1}, \frac{\frac{x_2+x_3}{x_1}+1}{x_2}, x_3\right), \begin{tikzcd} & 2 & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd} \right)$$

$\mathcal{A}_Q$ is generated by $x_1, x_2, x_3, \dfrac{x_2+x_3}{x_1}, \dfrac{\frac{x_2+x_3}{x_1}+1}{x_2}, \ldots$

Cluster Algebras

* exchange graph of $\mathcal{A}_Q$ is n-regular graph s.t.

vertices ~ seeds under equivalence of permuting the order of cluster variables and applying an isomorphism on the quivers

edges ~ mutations

Exercise a) prove that $\mu_k \circ \mu_k = 1$

b) compute the exchange graph of $\mathcal{A}_Q$ where $Q: 1 \to 2$

Cluster Algebras

① **Laurent Phenomenon** [Fomin-Zelevinsky 2001]

$$\mathcal{A}_Q \subset \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$$

② **Positivity** [Schiffler-Lee 2013]

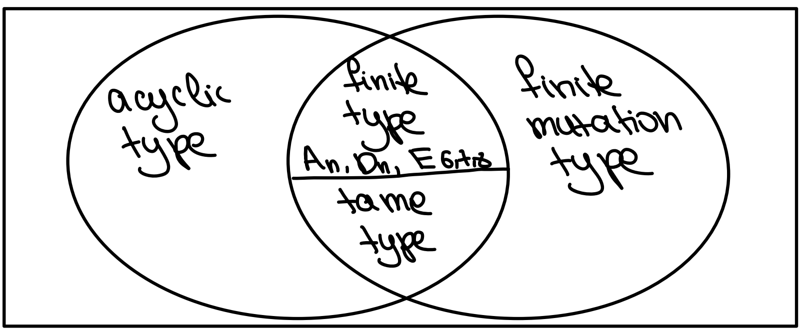
$$\{\text{cluster variables}\} \subset \mathbb{Z}_{\geq 0}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$$

Question: * can we find an explicit formula for cluster variables?

* what are the coefficients in a cluster variable counting?

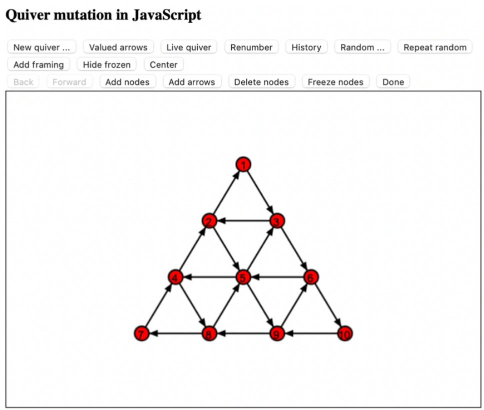
Cluster Algebras

* $\mathcal{A}_Q$ is of finite type if there are fin. many cluster variables
* $\mathcal{A}_Q$ is of finite mutation type if $Q' \overset{mut.}{\sim} Q$ is finite
* $\mathcal{A}_Q$ is acyclic if there exist $Q' \overset{mut}{\sim} Q$ that has no oriented cycles



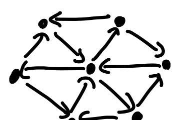
Cluster Algebras

* Keller's mutation applet



Exercise: a) explore the mutation applet

b) check that the following quiver is of acyclic type



Categorification

category $\mathcal{C}$		cluster algebra $\mathcal{A}_Q$
(certain) ind. objects M	$\xrightarrow[\text{map}]{\text{cluster character}}$	cluster variables x_M
cluster-tilting objects $M = \bigoplus_{i=1}^{n} M_i$		clusters $x_M = (x_{M_1}, \dots, x_{M_n}\}$
quiver of $\mathrm{End}_{\mathcal{C}}(M)$		quiver Q of x_M
categorical mutation		mutation

Thank you!