Categorification

Dimers Summer School

CUNY

August 14-18

Last time:

* quiver represenations

$$A = \Bbbk(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3) / \langle \alpha\beta \rangle$$

AR-quiver:

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[dr] & & & 1 \\
3 \arrow[ur] \arrow[rr, phantom, "\tau"] & & 2 \arrow[dr] \arrow[rr, phantom, "\tau"] & & \\
 & & & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[uur] &
\end{tikzcd}$$

Categorification

* cluster algebras $\qquad \mathcal{A}_Q \subset \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$$(\mathbf{x} = (x_1, x_2),\ 1 \to 2)$$

$$\xleftarrow{\mu_1} \left(\left(\frac{x_2+1}{x_1}, x_2 \right),\ 1 \leftarrow 2 \right) \qquad \xrightarrow{\mu_2} \left(\left(x_1, \frac{1+x_2}{x_2} \right),\ 1 \leftarrow 2 \right)$$

Overview

* Quivers with potential
* τ-tilting theory
* cluster character map
* cluster categories

Quivers with potential

[following Derksen-Weyman-Zelevinsky]

* Q quiver w/o loops (may contain 2-cycles)
* $\widehat{\Bbbk Q}$ - complete path algebra (allow paths of infinite length)
* W - (possibly infinite) linear combination of cycles in Q where we consider cycles up to cyclic equivalence $a_1 a_2 \cdots a_n \sim a_2 \cdots a_n a_1 \sim a_3 \cdots a_1 a_2 \sim \cdots$
* (Q, W) - quiver with potential

note if Q has no oriented cycles (acyclic)
$\widehat{\Bbbk Q} = \Bbbk Q$ and $W = 0$

Quivers with potential

* ∂_α : cycles $\longrightarrow \widehat{\Bbbk Q}$ cyclic derivative w.r.t. $\alpha \in Q_1$

$$\partial_\alpha(a_1 \cdots a_n) = \sum_{k : a_k = \alpha} a_{k+1} \cdots a_n a_1 \cdots a_{k-1}$$

* $J(W) = \overline{\langle \partial_\alpha W \mid \alpha \in Q_1 \rangle}$ ideal of $\widehat{\Bbbk Q}$

* $\mathrm{Jac}(Q, W) = \widehat{\Bbbk Q} / J(W)$ Jacobian algebra of a quiver with potential (Q, W)

* (Q, W) is Jacobi-finite if the Jacobian alg is finite dimensional

idea repr. theory of Jacobi-finite algebras* is closely related to combinatorics of cluster algebra

Quivers with potential

ex

$$Q: \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

cycles: $(\alpha\beta\gamma)^m$ for $m \geq 1$

let $W = \alpha\beta\gamma\alpha\beta\gamma$

then $\partial_\alpha(W) = \beta\gamma\alpha\beta\gamma + \beta\gamma\alpha\beta\gamma = 2\beta\gamma\alpha\beta\gamma$

$\partial_\beta(W) = 2\gamma\alpha\beta\gamma\alpha$

$\partial_\gamma(W) = 2\alpha\beta\gamma\alpha\beta$

$\mathrm{Jac}(Q,W) = \widehat{\Bbbk Q}/J(W)$ has basis of paths of length ≤ 4 if $\mathrm{char}\,\Bbbk \neq 2$

(Q,W) is Jacobi-finite

Quivers with potential

(Q,W) - quiver with potential

let $k \in Q_0$ not in a 2-cycle and define mutation

* $\widetilde{\mu}_k(Q)$ is obtained as follows:
  - every path $i \xrightarrow{\alpha} k \xrightarrow{\beta} j$ yields $i \xrightarrow{[\alpha\beta]} j$
  - replace every arrow $i \xrightarrow{\alpha} k$ by $i \xleftarrow{\alpha^*} k$

    $k \xrightarrow{\beta} j$ by $k \xleftarrow{\beta^*} j$

* $\widetilde{\mu}_k(W) = [W] + \Delta_k$

$$Q: \; i \xrightarrow{\alpha} k \xrightarrow{\beta} j$$

$$\mu_k(Q): \; \begin{tikzcd} i \arrow[rr, bend right, "{[\alpha\beta]}"'] & k \arrow[l, "\alpha^*"'] & j \arrow[l, "\beta^*"'] \end{tikzcd}$$

$$\Delta_k = \sum_{\substack{\text{all paths} \\ \text{through } k}} [\alpha\beta]\beta^*\alpha^*$$

$[W]$ obtained from W by $\alpha\beta \rightsquigarrow [\alpha\beta]$

<u>note</u> $\widetilde{\mu}_k(W)$ may contain 2-cycles, then if possible replace $\widetilde{\mu}_k(W)$ by an equivalent potential $\mu_k(W)$ s.t. $\mathrm{Jac}(\widetilde{\mu}_k(Q), \widetilde{\mu}_k(W)) \simeq \mathrm{Jac}(\mu_k(Q), \mu_k(W))$

Quivers with potential

* (Q, W) is nondegenerate if for any sequence of mutations μ, the resulting $\mu(Q, W)$ is 2-acyclic up to equivalence

* [DWZ]: if $\Bbbk$ is uncountable then there exists a nondegenerate potential for any 2-acyclic quiver Q

* (Q, W) is rigid if every cycle in Q is cyclically equivalent to an element of $J(W)$

* [DWZ] rigid $\Rightarrow$ nondegenerate

Quivers with potential

ex

$$Q: \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

cycles: $(\alpha\beta\gamma)^m$ for $m \geq 1$

* let $W = \alpha\beta\gamma\alpha\beta\gamma$, is (Q, W) rigid ?
* let $W' = \alpha\beta\gamma$, is (Q, W') rigid ? $J(W) = \overline{\langle \alpha\beta, \beta\gamma, \gamma\alpha \rangle}$

compute $\mu_1(Q, W')$

$$\tilde{\mu}_1(Q): \begin{tikzcd} & 2 \arrow[dl, "\alpha^*"'] \arrow[dr, shift left, "\beta"] & \\ 1 \arrow[rr, "\gamma^*"'] & & 3 \arrow[ul, shift left, "{[\gamma\alpha]}"] \end{tikzcd}$$

$$\tilde{\mu}_1(W') = \underbrace{[\gamma\alpha]\beta}_{[W]} + \underbrace{[\gamma\alpha]\alpha^*\gamma^*}_{\Delta_1}$$

$$J(\tilde{\mu}_1(W')) = \langle \underbrace{\beta + \alpha^*\gamma^*}_{\partial_{[\gamma\alpha]}}, \underbrace{[\gamma\alpha]}_{\partial\beta}, \underbrace{\gamma^*[\gamma\alpha]}_{\partial\alpha^*}, \underbrace{[\gamma\alpha]\alpha^*}_{\partial\gamma^*} \rangle$$

$$Jac(\tilde{\mu}_1(Q), \tilde{\mu}_1(W')) \cong Jac\left(\begin{tikzcd} & 2 \arrow[dl] \\ 1 \arrow[r] & 3 \end{tikzcd}, 0 \right) = \mathbb{k}\left(\begin{tikzcd} & 2 \arrow[dl] \\ 1 \arrow[r] & 3 \end{tikzcd} \right)$$

Quivers with potential

Exercise:

Let $Q =$
$$\begin{tikzcd} & 2 \arrow[dr, shift left, "\beta_1"] \arrow[dr, shift right, "\beta_2"'] & \\ 1 \arrow[ur, shift left, "\alpha_1"] \arrow[ur, shift right, "\alpha_2"'] & & 3 \arrow[ll, shift right, "\gamma_2"'] \arrow[ll, shift left, "\gamma_1"] \end{tikzcd}$$
$$W = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2$$

a) show that (Q, W) is not rigid

b) show that (Q, W) is nondegenerate

Quivers with potential

* mutation of (Q, W) induces a map on the associated module categories

$$\begin{array}{ccc} \operatorname{mod} \operatorname{Jac}(Q, W) & \xrightarrow{\mu_k} & \operatorname{mod} \operatorname{Jac}(\mu_k(Q), \mu_k(W)) \\ M & \longmapsto & \mu_k(M) \end{array}$$

that yields a bijection b/w indecomposable modules except for the simple module $S(k)$

$$M: \quad \begin{tikzcd} M_i \arrow[dr, "\varphi_\alpha"] & & M_j \\ & M_k \arrow[ur, "\varphi_\beta"] & \end{tikzcd} \quad \overset{\mu_k}{\rightsquigarrow} \quad \begin{tikzcd} M_i \arrow[rr, bend left, "\varphi_\beta \varphi_\alpha = \varphi_{[\alpha\beta]}"] & & M_j \arrow[dl, "\varphi_\beta^*"] \\ & M_k^* \arrow[ul, "\varphi_\alpha^*"] & \end{tikzcd}$$

Quivers with potential

fin. dim.

mod Jac(Q,W)

(reachable) indecomposable
τ-rigid modules M

support τ-tilting
pairs

$$\Big(\underbrace{\bigoplus_{i=1}^{t} M_i}_{\tau\text{-rigid modules}},\ \underbrace{\bigoplus_{j=t+1}^{n} P(j)}_{\text{projectives}}\Big)$$

mutation of
support τ-tilt. pairs

cluster alg of Q

non-initial
cluster variables x_M

clusters

$$(\underbrace{x_{M_1},\dots,x_{M_t}}_{\text{non-initial}},\underbrace{x_{t+1},\dots x_n}_{\text{initial}})$$

mutation

Quivers with potential

$$\begin{tikzcd}
\mathrm{Jac}(Q,W) \arrow[rr, "\text{cluster character}"] \arrow[d, "\mu_k(M)"'] & & \mathcal{A}_Q \arrow[d, "\cong"] \\
\mathrm{Jac}(\mu_k(Q),\mu_k(W)) \arrow[rr, "\text{cluster character}"] \arrow[urr, phantom, "\circlearrowright"] & & \mathcal{A}_{\mu_k(Q)}
\end{tikzcd}$$

Jac(Q,W): support τ-tilting theory

$\mathcal{A}_Q$: cluster combinatorics

Jac($\mu_k(Q),\mu_k(W)$): support τ-tilt. theory

$\mathcal{A}_{\mu_k(Q)}$: cluster combinatorics

*(Q,W) - Jacobi-finite and nondegenerate

τ- tilting Theory

[Adachi-Iyama-Reiten]

* Λ- fin. dim algebra
* $M \in \operatorname{mod}\Lambda$ is τ-rigid if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$
* (M, P) is support τ-tilting pair if
  - M is τ-rigid
  - $\operatorname{Hom}(P, M) = 0$ $\quad P$ - projective $\quad$ ($\operatorname{Hom}(P(i), M) = 0$ iff $M_i = 0$)
  - $M \oplus P$ has n indecomposable summands

* given an ind. summand X of $M \oplus P$ there exists a unique support τ-tilt. pair $\mu_X(M, P)$ obtained by replacing X with Y with $X \not\cong Y$.

τ-tilting Theory

ex $Q: 1 \to 2$

$\Bbbk Q = \mathrm{Jac}(Q, 0)$

AR quiver:

$$\begin{tikzcd}[column sep=small]
 & \underset{\Bbbk \to \Bbbk}{P(1)} \arrow[dr] & \\
\underset{0 \to \Bbbk}{P(2)} \arrow[ur] & \tau & \underset{\Bbbk \to 0}{S(1)} \arrow[ll, dashed]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
 & (P(1)\oplus P(2), 0) \arrow[dl, dash] \arrow[ddr, dash] & \\
(P(1)\oplus S(1), 0) \arrow[d, dash] & & \\
(S(1), P(2)) \arrow[dr, dash] & & (P(2), P(1)) \arrow[dl, dash] \\
 & (0, P(1)\oplus P(2)) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & \left(x', \frac{x_1+1}{x_2}\right) \arrow[dl, dash] \arrow[ddr, dash] & \\
\left(x', \frac{1+x_2}{x_1}\right) \arrow[d, dash] & & \\
\left(\frac{1+x_2}{x_1}, x_2\right) \arrow[dr, dash] & & \left(\frac{x_1+1}{x_2}, x_1\right) \arrow[dl, dash] \\
 & (x_1, x_2) &
\end{tikzcd}$$

support τ-til. poset of $\Bbbk Q$ **exchange graph of $\mathcal{A}_Q$**

Cluster Character

* $\varphi : \operatorname{mod}\Lambda \longrightarrow \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$
  $M \longmapsto \varphi(M)$

$$\boxed{\varphi(M) = x^{\operatorname{ind}(M)} \cdot \sum_{\underline{e} \in \mathbb{Z}^n_{\geq 0}} \chi(\operatorname{Gr}_{\underline{e}}(M)) \prod_{i=1}^{n} x_i^{\langle S(i), \underline{e} \rangle}}$$

$\operatorname{ind}(M)$:

$0 \to M \hookrightarrow \bigoplus I(i) \to \bigoplus I(j)$ minimal injective resolution of M

$\operatorname{ind}(M) = \sum e_j - \sum e_i$

$\chi(\operatorname{Gr}_{\underline{e}}(M))$: Euler characteristic of submodule Grassmannian of M of dimension $\underline{e}$

$\langle S(i), e_j \rangle$: $\dim \operatorname{Hom}(S(i), S(j)) - \dim \operatorname{Ext}^1(S(i), S(j))$
$- \dim \operatorname{Hom}(S(j), S(i)) + \dim \operatorname{Ext}^1(S(j), S(i))$

Cluster Character

ex $Q: 1 \to 2$

$\Bbbk Q = \mathrm{Jac}(Q, 0)$

AR quiver:

$$\begin{tikzcd}[column sep=small]
 & \underset{\Bbbk \to \Bbbk}{P(1) = I(2)} \arrow[dr] & \\
\underset{0 \to \Bbbk}{P(2)} \arrow[ur] & & \underset{\Bbbk \to 0}{S(1) = I(1)} \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

compute $\varphi(P(2))$?

$\mathrm{ind}(P(2))$: $0 \to P(2) \hookrightarrow I(2) \to I(1)$

$\mathrm{ind}\, P(2) = e_1 - e_2 = (1, -1)$

$\chi(Gr_{\underline{e}}(P(2)))$: only two subrepresentations

$0 \rightsquigarrow \underline{e} = (0,0) \rightsquigarrow \chi(Gr_{\underline{e}}(P(2))) = 1$

$P(2) \rightsquigarrow \underline{e} = (0,1) \rightsquigarrow \chi(Gr_{\underline{e}}(P(2))) = 1$

$\langle S(1), e_2 \rangle = (\#\text{arrows } 2 \to 1) - (\#\text{arrows } 1 \to 2) = -1$

$\langle S(2), e_2 \rangle = 0$

Cluster Character

ex $Q: 1 \to 2$

$\Bbbk Q = \mathrm{Jac}(Q, 0)$

AR quiver:

$$\begin{tikzcd}
 & \underset{\Bbbk \to \Bbbk}{P(1)} = I(2) \arrow[dr] & \\
\underset{0 \to \Bbbk}{P(2)} \arrow[ur] & & \underset{\Bbbk \to 0}{S(1)} = I(1) \arrow[ll, dashed]
\end{tikzcd}$$

compute $\varphi(P(2))$?

* $\mathrm{ind}\, P(2) = e_1 - e_2 = (1, -1)$
* $\chi(\mathrm{Gr}_{\underline{e}}(P(2)))$: only two subrepresentations

  $0 \rightsquigarrow \underline{e} = (0,0) \rightsquigarrow \chi(\mathrm{Gr}_{\underline{e}}(P(2))) = 1$

  $P(2) \rightsquigarrow \underline{e} = (0,1) \rightsquigarrow \chi(\mathrm{Gr}_{\underline{e}}(P(2))) = 1$
* $\langle S(1), e_2 \rangle = (\#\text{arrows } 2 \to 1) - (\#\text{arrows } 1 \to 2) = -1$

  $\langle S(2), e_2 \rangle = 0$
* $\varphi(P(2)) = x_1^{1} x_2^{-1}(1 + x_1^{-1}) = \dfrac{x_1 + 1}{x_2}$

Cluster Character

Exercise: Let $Q: 1 \longrightarrow 2 \longrightarrow 3$

$\mathrm{Jac}(Q,0) = \Bbbk Q$

Let $X = (0, P(1) \oplus P(2) \oplus P(3)) = \left(0, \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3\right)$ be the support τ-tilting pair in $\Bbbk Q$

a) calculate the three support τ-tilting pairs obtained from X via a single mutation i.e. calculate $\mu_{P(i)}(X)$ for $i = 1,2,3$

b) calculate $\underbrace{\varphi}_{\text{cluster character}}(\mu_{P(i)}(X))$ for $i=1,2,3$

c) show that $\varphi(\underbrace{\mu_{P(i)}(X)}_{\text{support } \tau\text{-tilting mutation}}) = \underbrace{\mu_i}_{\text{mutation in } \mathcal{A}_Q}(\varphi(X)) = \mu_i(\underbrace{(x_1, x_2, x_3)}_{\text{initial seed}})$

Cluster Character

Exercise*

Let (Q,W) be a Jacobi-finite quiver with potential w/o 2-cycles

a) describe support τ-tilting pairs obtained from $X = \left(0, \bigoplus_{i \in Q_0} P(i)\right)$ via a single mutation

b) show that $\varphi(\mu_{P(k)}(X)) = \mu_k(\varphi(X))$

i.e. the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mu_{P(k)}"'] & (x_1, \ldots, x_n) \arrow[d, "\mu_k"] \\ \mu_{P(k)}(X) \arrow[r, "\varphi"] & \mu_k(x_1, \ldots, x_n) \end{tikzcd}$$

initial cluster

Cluster Category

* (Q,W) Jacobi-finite quiver with non-degenerate potential

* $\mathcal{C}_{(Q,W)}$ cluster category [Buan-Marsh-Reineke-Reiten-Todorov, Amiot]
triangulated, Hom-finite and 2-Calabi-Yau

* $T \in \mathcal{C}_{(Q,W)}$ cluster tilting object (c.t.o)
i.e. $T = \bigoplus_{i=1}^{n} T_i$ and $\mathrm{Ext}^1(T,T)=0$
s.t. $\mathrm{End}_{\mathcal{C}}(T) \cong \mathrm{Jac}(Q,W)$

* $$F_T : \mathcal{C}_{(Q,W)} \longrightarrow \mathrm{mod}\,\mathrm{Jac}(Q,W)$$
$$X \longmapsto \mathrm{Hom}(T,X)$$
$$\mathcal{C}_{(Q,W)}/T[1] \cong \mathrm{mod}\,\mathrm{Jac}(Q,W)$$

Cluster Category

* $\mathcal{C}_{(Q,W)}$ - cluster category

* mutation for c.t.o T at T_i [Iyama-Yoshino]

  then $\mu_{T_i}(T) = T \setminus T_i \oplus T_i^*$

*
$$\begin{tikzcd}
T \arrow[r, "F_T"] \arrow[d, "\text{IY mutation}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{End}(T) \arrow[d, "\text{DWZ mutation}"] \\
T' \arrow[r, "F_{T'}"'] & \mathrm{End}(T')
\end{tikzcd}$$

Cluster Category

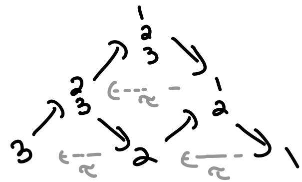
$\mathcal{C}_{(Q,W)}$ $\xrightarrow[\text{"}\varphi\circ F_T\text{"}]{\varphi_e}$ $\mathcal{A}_Q$

summands of c.t.o T_i (reachable)	cluster variables x_i
c.t.o T (reachable)	clusters $\mathbf{x}_T$
$\mathrm{End}(T)$	quiver of $\mathbf{x}_T$
exchange triangles $T_i^* \to T_A \to T_i$ $T_i \to T_B \to T_i^*$	exchange relations $x_i x_i^* = x_A + x_B$

Cluster Category

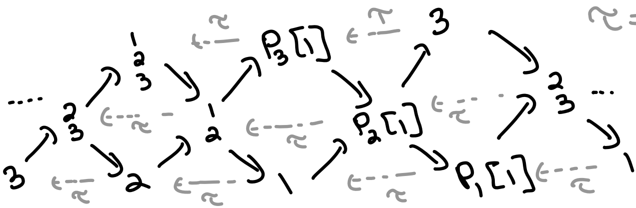
ex $Q = 1 \to 2 \to 3$

$\operatorname{mod} \Bbbk Q$



$\tau = [1]$

$\mathcal{C}_{(Q,0)}$



$T = 3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \rightsquigarrow \operatorname{End}(T) = 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\\3\end{smallmatrix} \cong \Bbbk(1 \to 2 \to 3)$

$\mu_2(T) = 3 \oplus 1 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \rightsquigarrow \operatorname{End}(\mu_2(T)) \cong$
$$\begin{tikzcd} 3 \arrow[r, "\alpha"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r, "\beta"] & 1 \arrow[ll, bend left, "\gamma"] \end{tikzcd}$$
with relations $\langle \alpha\beta, \beta\gamma, \gamma\alpha \rangle$

$$\cong \operatorname{Jac}\left(\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \arrow[ll, bend left, "\gamma"] \end{tikzcd}, \alpha\beta\gamma\right)$$

Summary

mutate (Q,W)	DWZ mutation	mutate Q
$\mathcal{C}_{(Q,W)}$	$\mathrm{Jac}(Q,W)$	$\mathcal{A}_Q$
ind. rigid (reachable) objects	ind τ-rigid modules together with $(0, P(i))$	cluster variables
c.t.o. T	support τ-tilt. pairs	clusters x_T
$\mathrm{End}(T)$	—	quiver of x_T
IY-mutation	mutation	mutation

Thank you !