# Categorification

Dimers Summer School

CUNY

August 14 - 18

# Overview

* Circle algebra $B_{k,n}$ and categorification
* Dimer models
* Preprojective algebra
* Perfect matching modules and the twist map

# Grassmannians

* $Gr(k,n)$ Grassmannian of $k$-planes in $\mathbb{C}^n$
* $P_I$ - Plücker coordinate $I \in \binom{[n]}{k}$
* $\mathbb{C}[Gr(k,n)]$ is a cluster algebra [Scott]
* $\mathbb{C}[Gr(k,n)] = \langle P_I \mid I \in \binom{[n]}{k} \rangle \Big/ \left(\text{Plücker relations}\right)$

$$\{P_I\} \subset \{\text{cluster variables}\}$$

$$\left\{\begin{array}{c}\text{seeds consisting} \\ \text{of Plücker} \\ \text{coordinates}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{plabic} \\ \text{graphs}\end{array}\right\}$$

# Categorification

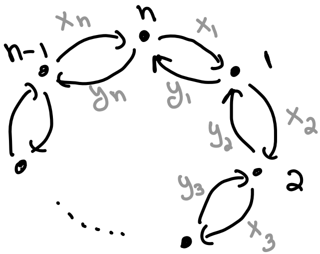
* [Geiss-Leclerc-Schröer] provide categorification of cluster structure on $\mathbb{C}[Gr(k,n)]$ using module category of preprojective algebra here $p_{12..k}=1$ and it corresponds to the zero module

* [Jensen-King-Su] give a different categorical model for cluster structure on $\mathbb{C}[Gr(k,n)]$ using module category of circle algebra which is more uniform but uses infinite dim. modules

# Circle algebra

* $B_{k,n}$ is given by a quiver with relations

$Q_{k,n}$:




$xy = yx$

$x^k = y^{n-k}$

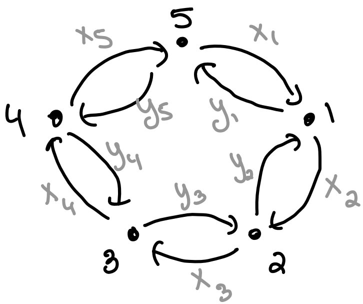
$$B_{k,n} = \widehat{\mathbb{C}Q_{k,n}} \Big/ \left\langle \begin{array}{l} xy-yx, \\ x^k - y^{n-k} \end{array} \right\rangle$$

* $B_{k,n}$ is infinite dimensional why?

* $B_{k,n}$ is Krull-Schmidt $\Rightarrow$ ind. projectives $\Leftrightarrow$ vertices of $Q_{k,n}$

# Circle algebra

ex $k=2$, $n=5$

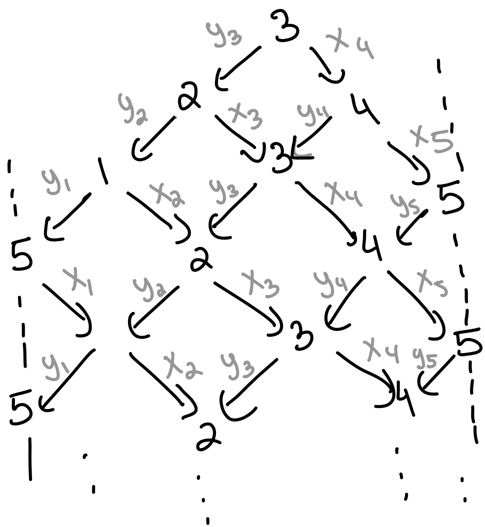



$x^2 = y^3$

$xy = yx$

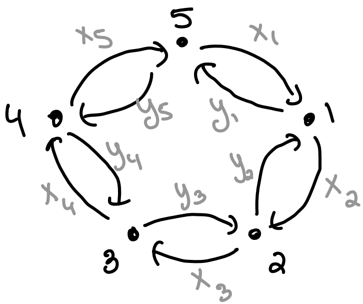
compute $P(3)$ ?

basis of $P(3)$ ~ paths starting at 3



# Circle algebra

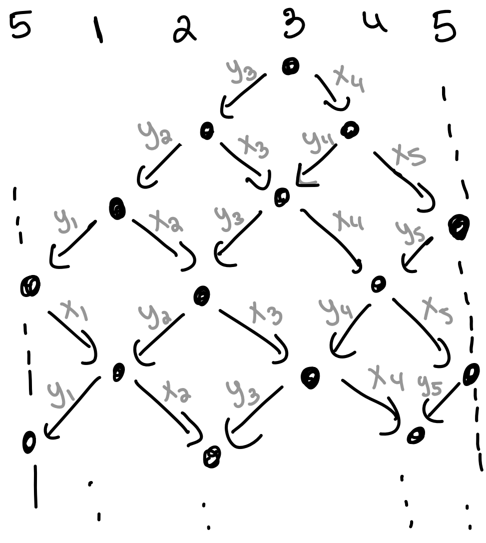
ex $k=2$, $n=5$




$$x^2 = y^3$$
$$xy = yx$$

compute $P(3)$ ?

basis of $P(3) \rightsquigarrow$ paths starting at 3



# Circle algebra

* $B_{k,n}$ is Gorenstein

* $CM(B_{k,n}) = \{ M \in \mathrm{Mod}\, B_{k,n} \mid \mathrm{Ext}^i(M, \mathrm{Proj}) = 0 \}$
  is Frobenius category with projective-injective objects being projective $B_{k,n}$-modules

* $CM(B_{k,n})$ provides categorification of cluster structure on $\mathbb{C}[Gr(k,n)]$

| $CM(B_{k,n})$ | $\mathbb{C}[Gr(k,n)]$ |
| --- | --- |
| ind. rigid modules $M$ i.e. $\mathrm{Ext}^1(M,M)=0$ | cluster variables $x_M$ |
| cluster-tilting modules | clusters |
| projective-injective | frozens |
| rank 1 modules | Plücker coordinates |

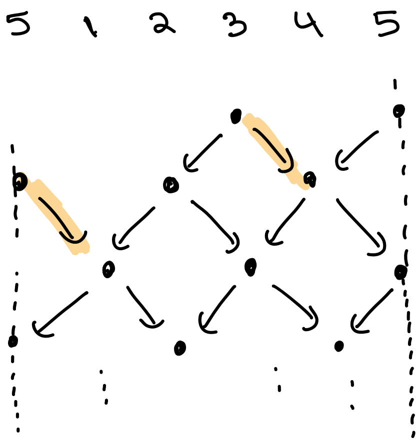
# Modules in $CM(B_{k,n})$

* what do rank 1 modules look like?

$$M_I \leftrightarrow I \in \binom{[n]}{k} \leadsto P_I$$

**Prop** $\mathrm{Ext}^1(M_I, M_J) = 0$ iff $I$ and $J$ are noncrossing

ex $k=2,\ n=5 \qquad I = 35$




$= M_{35}$

# Modules in $CM(B_{k,n})$

* What do modules beyound rank 1 look like?

**Prop** $M \in CM(B_{k,n})$ has a filtration by rank 1 modules.

$$\begin{tikzcd}
\cdots \arrow[r, phantom, "\subset"] & M_2 \arrow[r, phantom, "\subset"] \arrow[d, two heads] & M_1 \arrow[r, phantom, "\subset"] \arrow[d, two heads] & M \arrow[d, two heads] \\
& M_2/M_3 = M_{I_3} & M_1/M_2 = M_{I_2} & M/M_1 = M_{I_1}
\end{tikzcd}$$

$M$ has filtration by $\ldots, M_{I_3}, M_{I_2}, M_{I_1}$

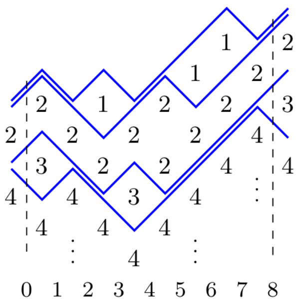
We say $M$ has **profile** $\dfrac{I_1}{\dfrac{I_2}{\dfrac{I_3}{\vdots}}}$ and rank the # of $I$'s in its profile

# Modules in CM($B_{k,n}$)

**ex** M - rank 4 module

137
125
124
238

$\underbrace{\qquad}$ profile of M




$\underbrace{\qquad\qquad}$ contours of M

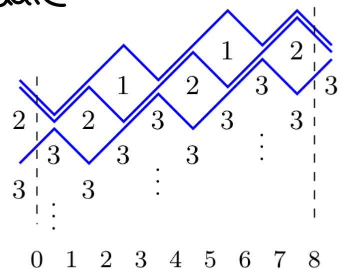
$\Rightarrow$ dimension of M

in Gr(3, 8)

**ex** M - rigid rank 3 module

368
258
147

$\underbrace{\qquad}$ profile of M




$\underbrace{\qquad\qquad}$ contour of M

in Gr(3, 8)

# Modules in $CM(B_{k,n})$

* profile does not determine the module $M$

* if $M$ is rigid then there is unique ind. (up to isomorphism) module with that profile

* if $M$ is indecomposable then its contours must be "close-packed" i.e. there is no unoriented path that goes around the cylinder between the contours

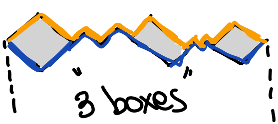
# Modules in $CM(B_{k,n})$

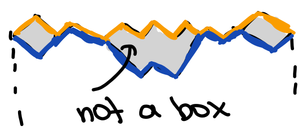
* want to understand rigid indecomposable modules of higher rank

* [Baur - Bogdanic - Garcia Elsener - Li]
  $M$ rank 2 module with profile $\frac{I}{J}$
  if the contours form exactly "3 boxes" it is rigid

<u>Conjecture</u> $M$ is rigid iff the contours form "3 boxes"

ex



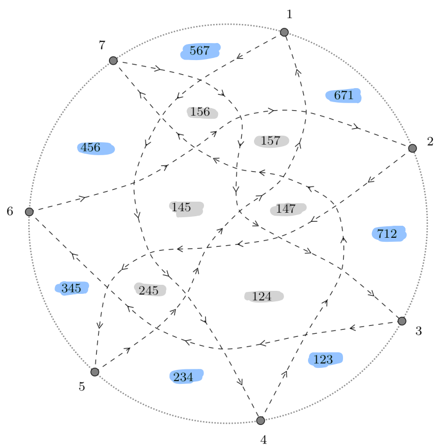



# Dimer Models

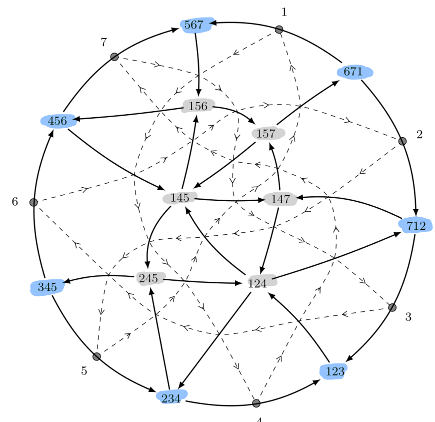
[Baur-King-Marsh]

recall: {Postnikov diagrams}/~ $\longleftrightarrow$ {clusters consisting of Plückers}

ex




Postnikov diagram D in Gr(3,7)




Quiver $Q_D$ of associated seed

# Dimer Models

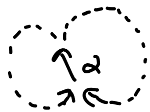
* Dimer algebra $A_D$:

$\mathbb{C}Q_D$ - path algebra of the quiver $Q_D$ coming from a Postnikov diagram

$$W = \sum \left\{ \begin{matrix} \text{clockwize} \\ \text{chordless} \\ \text{cycles} \end{matrix} \right\} - \sum \left\{ \begin{matrix} \text{cclockwize} \\ \text{chordless} \\ \text{cycles} \end{matrix} \right\}$$

$$I(W) = \left\langle \partial_\alpha W \;\middle|\; \begin{matrix} \alpha \text{ is not} \\ \text{a boundary} \\ \text{arrow} \end{matrix} \right\rangle$$

$$= \langle \, p = q \mid p \quad q \, \rangle$$



$$A_D = \mathbb{C}Q / I(W)$$

(can also consider the complete version)

# Dimer Models

* what is the connection b/w $CM(B_{k,n})$, combinatorics of Postnikov diagrams $D$, and dimer algebras $A_D$?

**Theorem** [BKM]

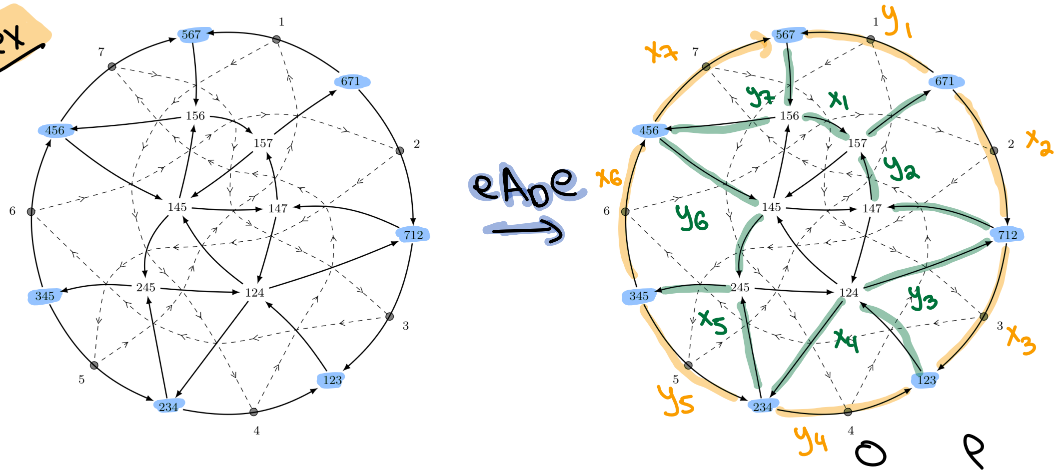
$$\begin{array}{c}\text{diagram } D \\ \text{with Plückers} \\ I_1, \dots, I_m\end{array} \xrightarrow{[JKS]} \begin{array}{c}\text{cluster tilting} \\ \text{module} \\ T_D = \bigoplus_{j=1}^{m} M_{I_j}\end{array} \rightsquigarrow \mathrm{End}_{B_{k,n}} T_D$$

then $A_D \cong \mathrm{End}_{B_{k,n}} T_D$

$e A_D e \cong B_{k,n}$ where $e = \sum_{i \text{ frozen}} e_i$

# Dimer Models

ex




* any path b/w the frozen vertices is equivalent to a path in x's and y's

* moreover $xy=yx$ and $x^k=y^{n-k}$

# Preprojective Algebra

[Geiss-Leclerc-Schröer]

* $Q$: $\begin{tikzcd} 1 \arrow[r, bend left, "x_1"] & 2 \arrow[l, bend left, "y_1"] \arrow[r, bend left, "x_2"] & 3 \arrow[l, bend left, "y_2"] \arrow[r, bend left, "x_3"] & \cdots \arrow[l, bend left, "y_3"] \arrow[r, bend left, "x_{n-1}"] & n-1 \arrow[l, bend left, "y_{n-1}"] \end{tikzcd}$

$\Pi_{n-1} = \mathbb{C}Q / \langle xy = yx \rangle$ Preprojective algebra

* $I(k)$ - injective $\Pi_{n-1}$ module at vertex $k$

* $\mathrm{Sub}(I(k))$ - submodules of $\bigoplus I(k)$

**Th** $\mathrm{Sub}(I(k))$ provides a categorification of cluster structure on $\mathbb{C}[Gr(k,n)]$

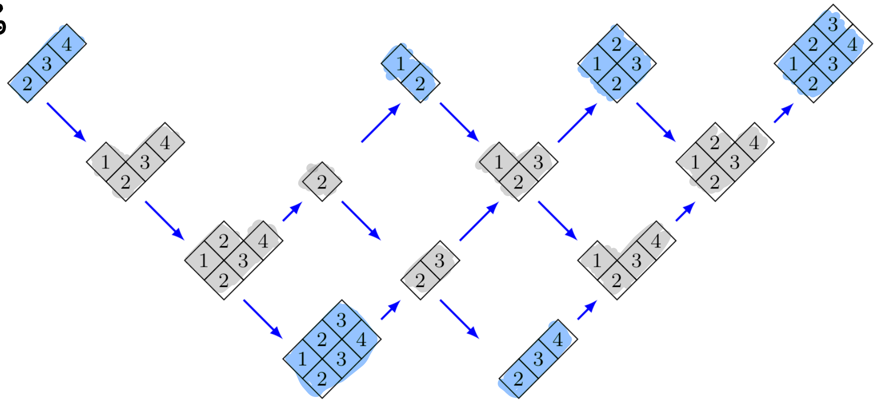
# Preprojective Algebra

ex $k=2,\ n=5$

$\Pi_4 :\ 1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4$

$I(2) = \begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}$

AR-quiver of $\mathrm{Sub}(I(2))$:



projective-injective objects of $\mathrm{Sub}(I(2))$

$\iff$ frozen cluster variables

# Preprojective Algebra

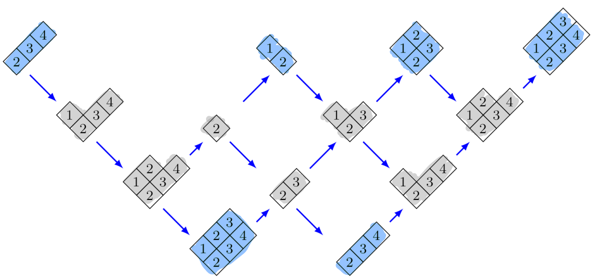
**ex** cluster alg of w/o frozens $\mathbb{C}[Gr(3,5)] \cong \mathcal{A}_{1\to 2}$

$\cong$

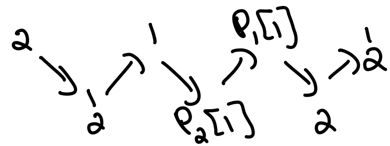
$\mathbb{C}[Gr(2,5)]$

AR-quiver of $Sub(I(2))$

AR-quiver of $\mathcal{C}_{(1\to 2,\, 0)}$



factor out by frozens $\longrightarrow$



# Preprojective Algebra

* $B_{k,n} / \langle e_n \rangle \cong \Pi_{n-1} / \langle x^k, y^{n-k} \rangle$

thus there is a functor

$$\begin{array}{ccc} \pi : \operatorname{mod} B_{k,n} & \longrightarrow & \operatorname{mod} \Pi_{n-1} / \langle x^k, y^{n-k} \rangle \\ \cup & & \cup \\ CM(B_{k,n}) & \overset{\pi}{\dashrightarrow} & \operatorname{Sub}(I(k)) \end{array}$$

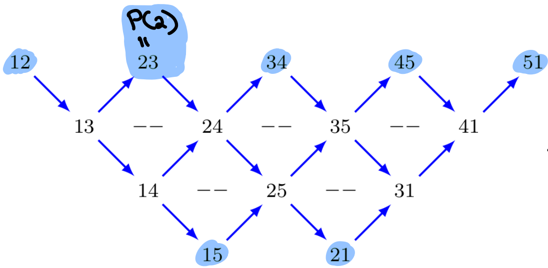
* [JKS] $\pi$ is almost an equivalence except

$\pi(P(n)) = 0$

# Preprojective Algebra

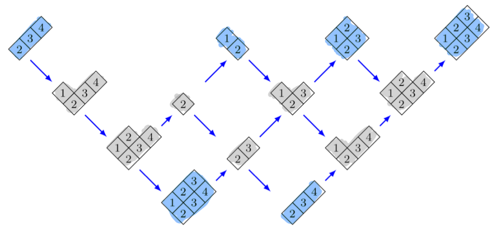
ex. $k=2,\ n=5$

AR-quiver of $CM(B_{2,5})$




$\xrightarrow{\pi}$

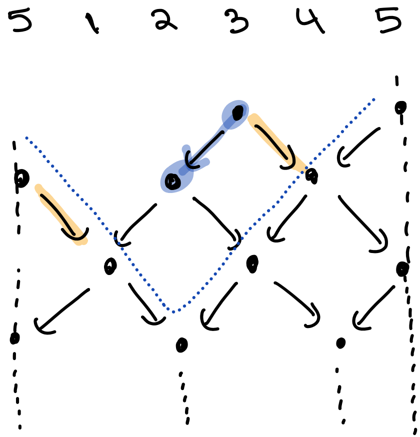
AR-quiver of $Sub(I(2))$



# Preprojective Algebra

ex $k=2,\ n=5$ $\qquad I = 35$

$M_{35} =$




remove $P(5)$ $\rightsquigarrow$ $\pi(M_{35}) = \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix}$

# Perfect Matching Modules

[Canakci-King-Pressland]

* $A_D$ - dimer algebra from a Postnikov diagram with quiver $Q$

* $\mu$ - perfect matching of $A_D$: i.e. a collection of arrows of $Q$, exactly one for each cycle

* $t = \sum_{i \in Q_0} t_i$ where $t_i$ - elementary cycle in $Q$ at $i$

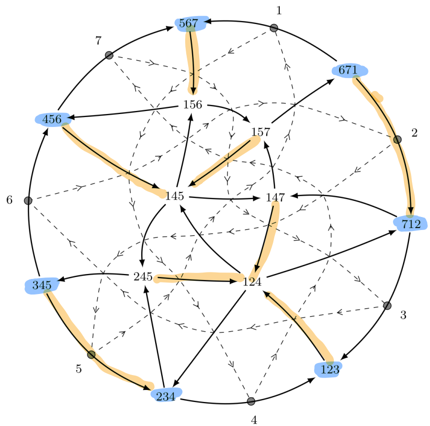
* $t \in$ center of $A_D$, so $A_D$ is a $\mathbb{Z}$-algebra

* $M_\mu$ - perfect matching module where at every vertex of $Q$ place a copy of $\mathbb{Z}$ for every arrow $\mathbb{Z} \xrightarrow{\varphi_\alpha} \mathbb{Z}$ $\varphi_\alpha = \begin{cases} 1 & \text{if } \alpha \notin \mu \\ t & \text{if } \alpha \in \mu \end{cases}$

# Perfect Matching Modules

* $\left\{\text{perfect matchings } \mu\right\} \longleftrightarrow \left\{\text{perfect match. modules } M_\mu\right\} \longleftrightarrow \left\{\text{rank one modules of } A_D\right\}$

ex




Th

$$\begin{tikzcd} P_I \arrow[r, "\text{Marsh-Scott twist map}"] & MS(P_I) \\ M_I \arrow[u, "\text{cluster character}"] \arrow[r] & \Omega^{\circ} M_I \arrow[u] \end{tikzcd}$$

kernel of some non-minimal proj-presentation of $M_I$

$\Omega^{\circ} M_I \cong \tau^{-1} M_I \oplus \text{Proj}$

Thank you !