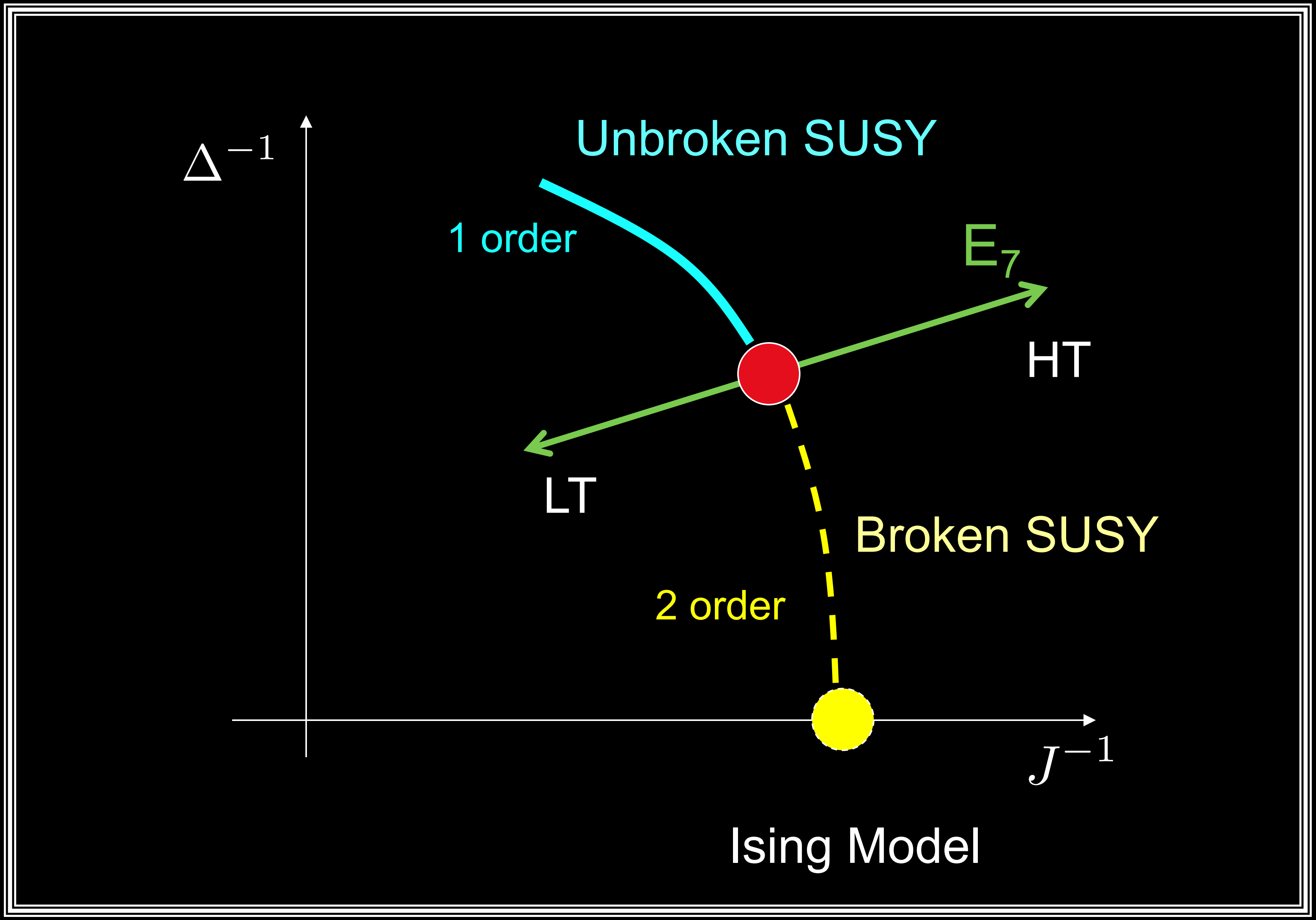
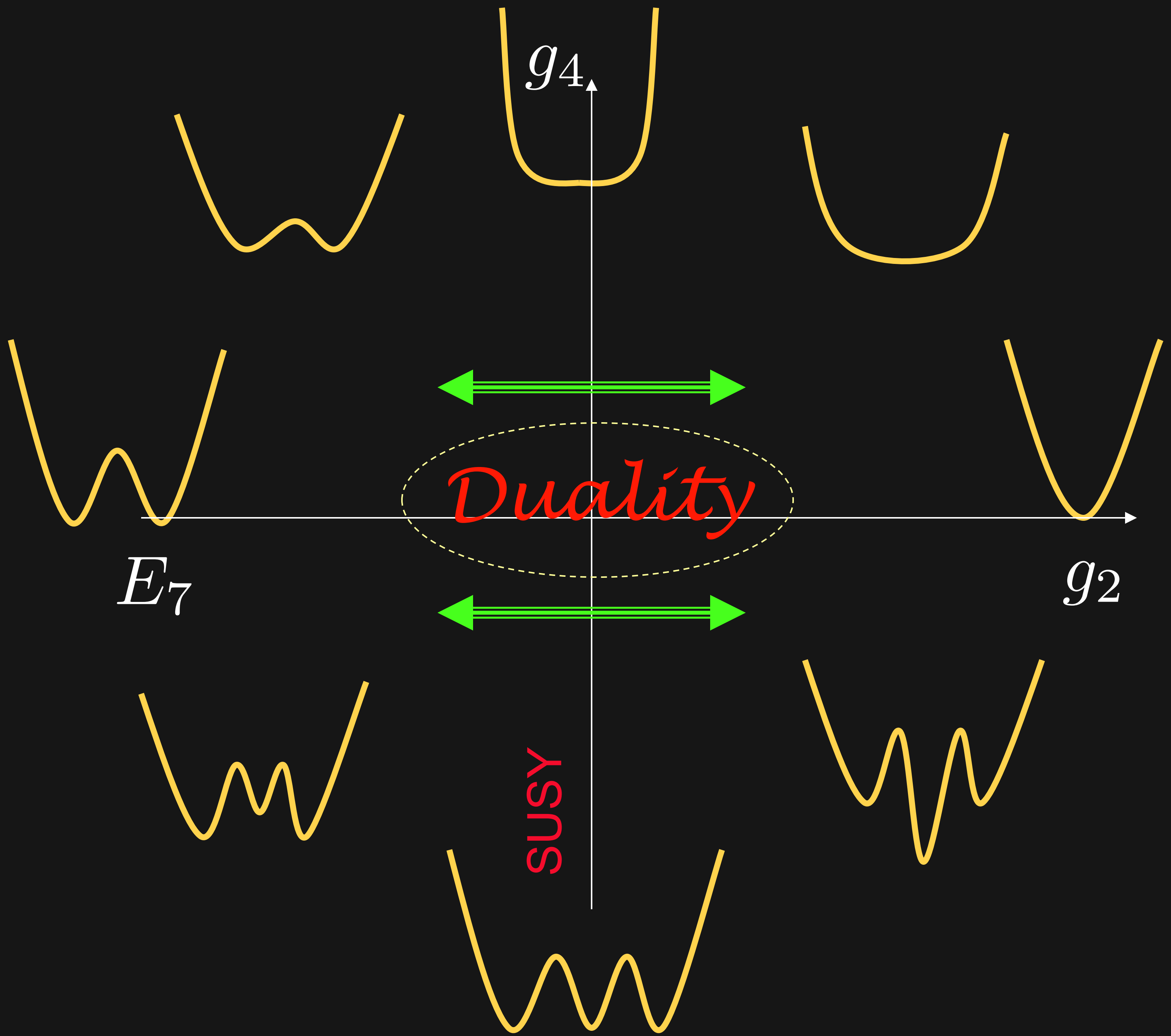
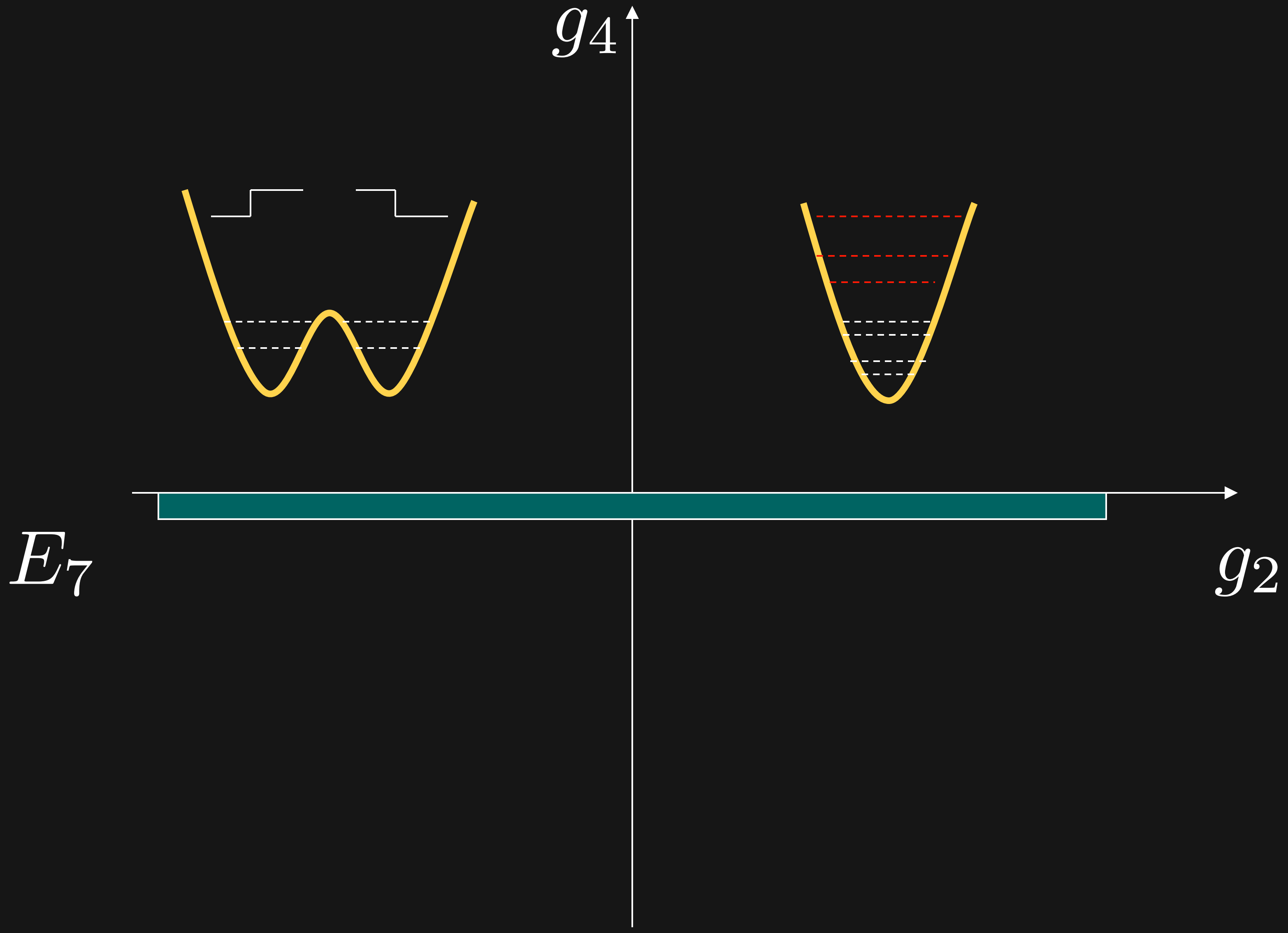
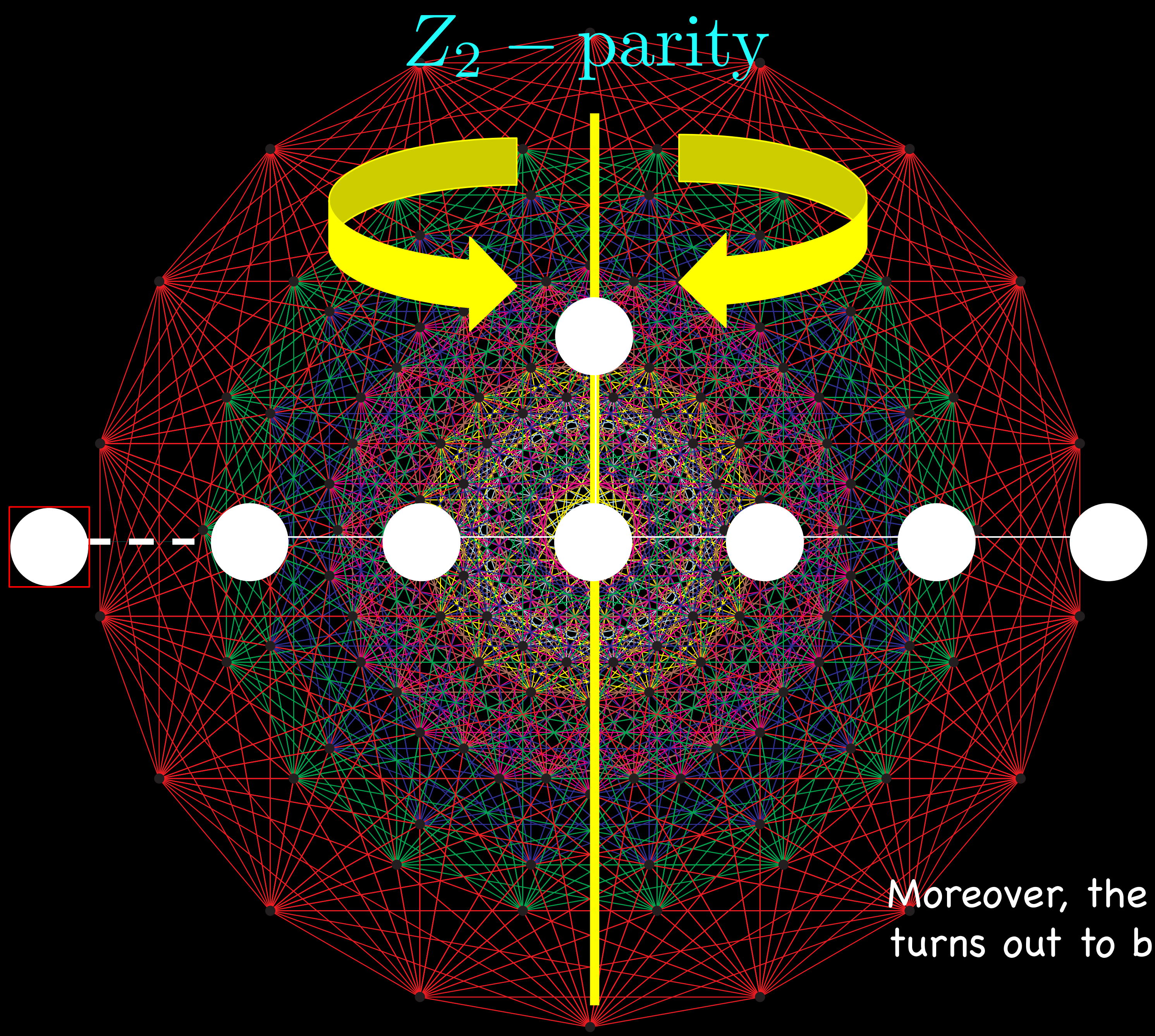
Phase diagram of TIM




g_4
Duality
E_7
g_2
SUSY


g_4
E_7
g_2



Moreover, the off-critical theory turns out to be integrable!

Thermal deformation (High temp.): E_7

(Christe-Mussardo, Nucl.Phys. B330)

There are conservation laws

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}$$

for the following values of the “spin” s

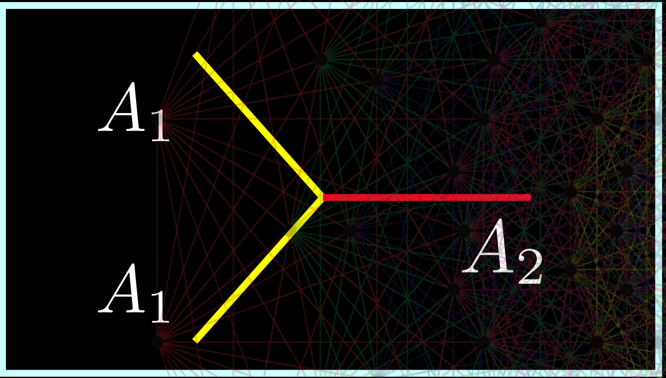
$$s = 1, 5, 7, 9, 11, 13, 17 \, (mod\, 18)$$

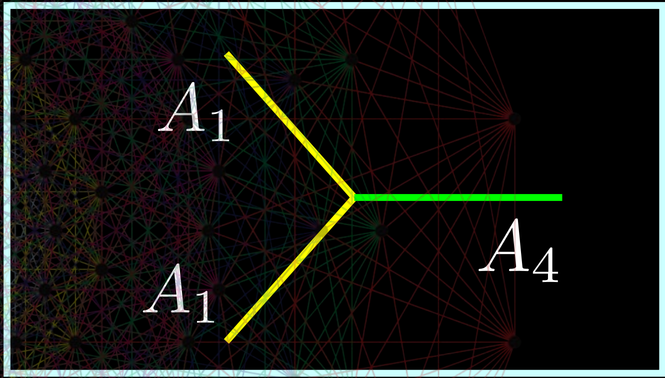
Coxeter Exponents of E_7

Thermal deformation (High temp.): E_7

(Christe-Mussardo, Nucl.Phys. B330)

The conservation laws are compatible with non-zero 3-particle couplings





with mass ratios

$$\frac{m_2}{m_1} = 2\cos\frac{5\pi}{18}$$

$$\frac{m_4}{m_1} = 2\cos\frac{\pi}{18}$$

The S-matrix of the fundamental particle is

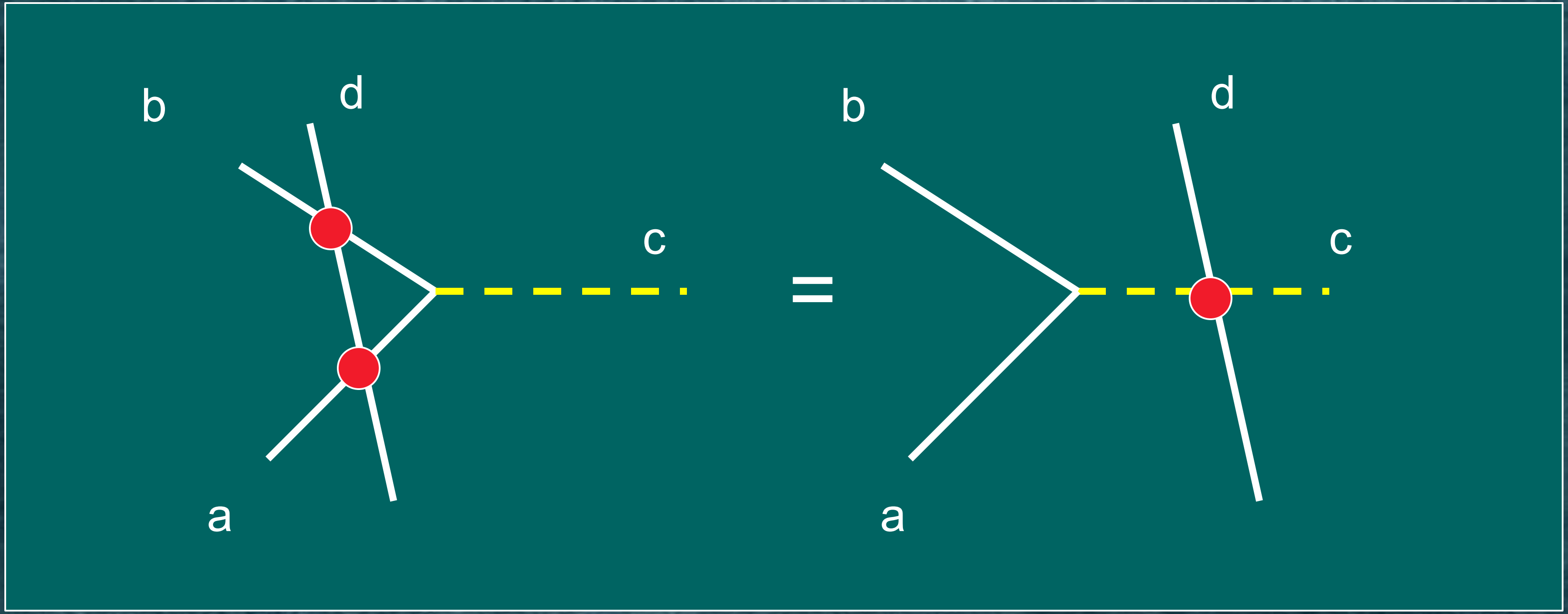
$$S_{11}(\beta) = -\frac{\tanh\frac{1}{2}\left(\beta + i\frac{5\pi}{9}\right)}{\tanh\frac{1}{2}\left(\beta - i\frac{5\pi}{9}\right)}\;\frac{\tanh\frac{1}{2}\left(\beta + i\frac{\pi}{9}\right)}{\tanh\frac{1}{2}\left(\beta - i\frac{\pi}{9}\right)}$$

A_2 A_4

The remaining S_{ab} are obtained by bootstrap

Bootstrap principle

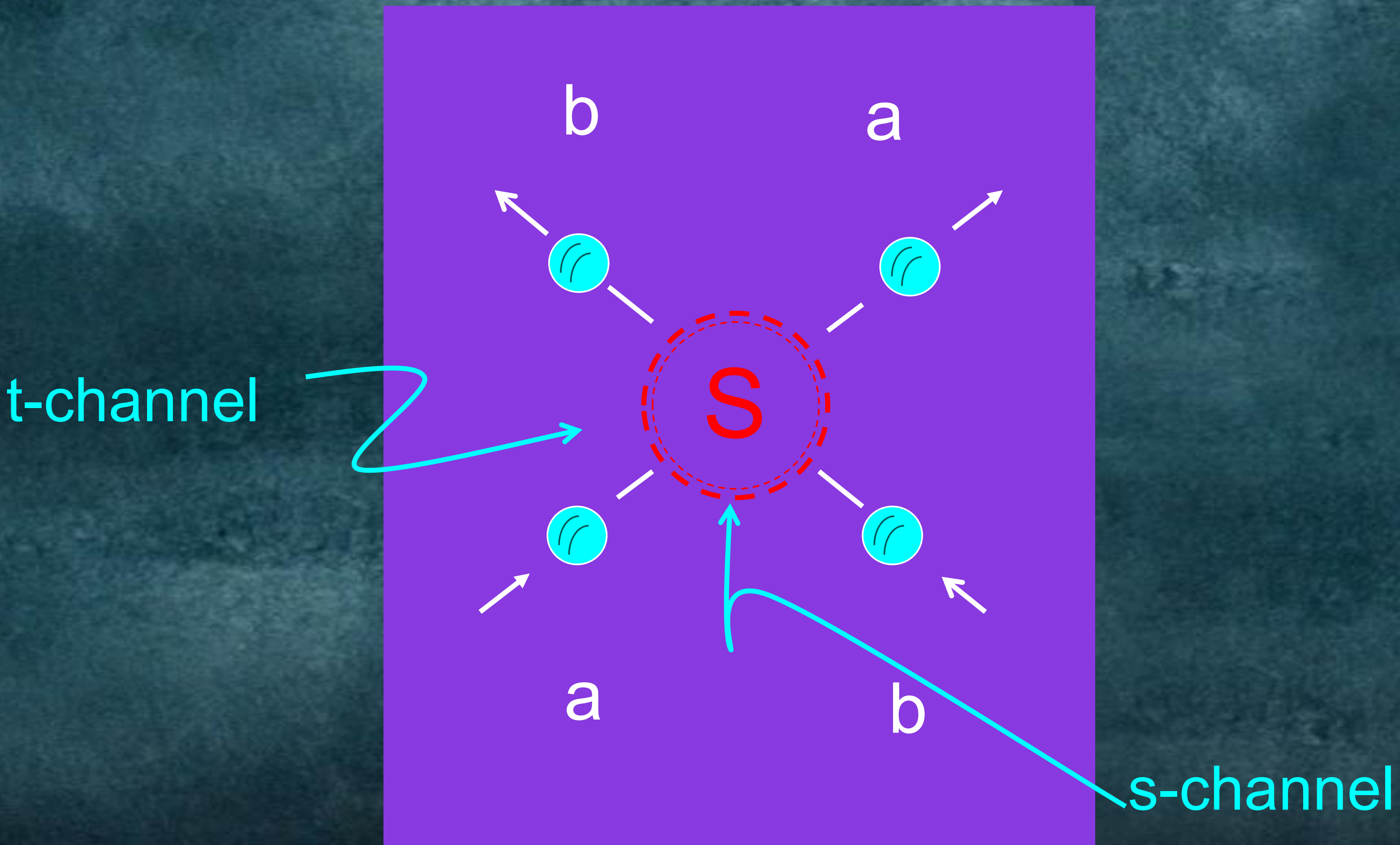
"All particles are equal but one is more equal than the others"



$$S_{cd}(\beta) = S_{ad}(\beta + i\bar{u}_{ac}^{b})\, S_{bd}(\beta - i\bar{u}_{bc}^{a})$$

Exact E_7 S-matrix

- 28 amplitudes $S_{ab}(\beta) \quad (a, b = 1, \ldots, 7)$



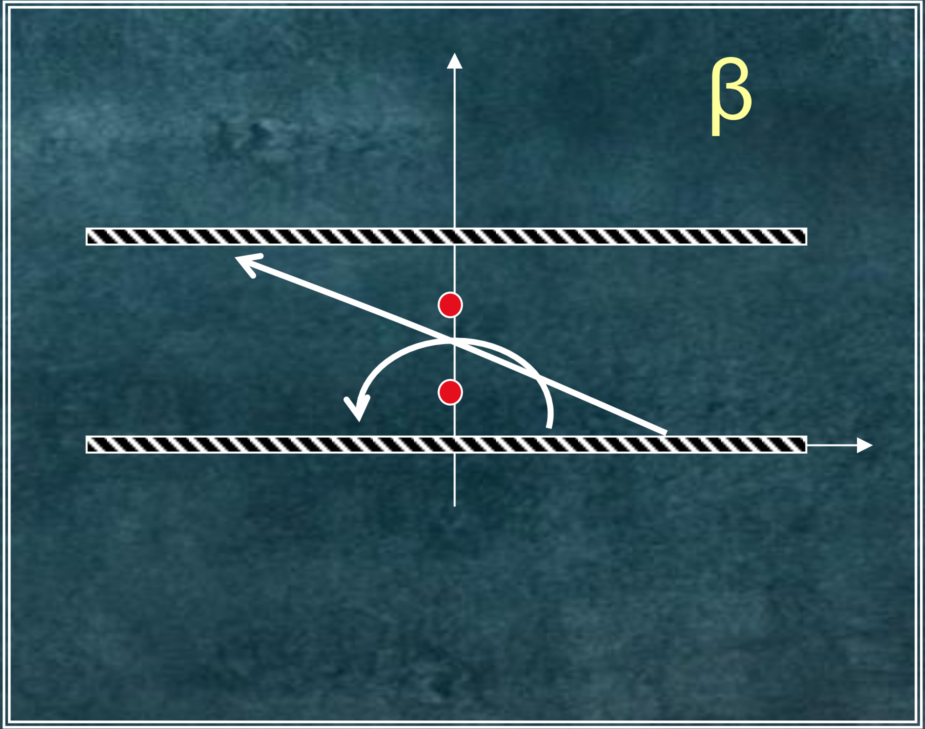
Exact S-matrix

- 28 amplitudes $S_{ab}(\beta) \quad (a, b = 1, \ldots, 7)$

- Each amplitude satisfies unitarity and crossing eqs.

$$S_{ab}(\beta) S_{ab}(-\beta) = 1$$

$$S_{ab}(i\pi - \beta) = S_{ab}(\beta)$$



Exact S-matrix

- 28 amplitudes $S_{ab}(\beta) \quad (a, b = 1, \ldots, 7)$

- Each amplitude satisfies unitarity and crossing eqs.

$$S_{ab}(\beta) S_{ab}(-\beta) = 1$$

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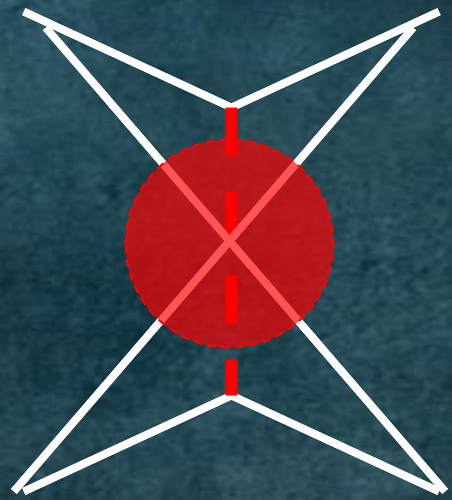
- Pole structure and bound states

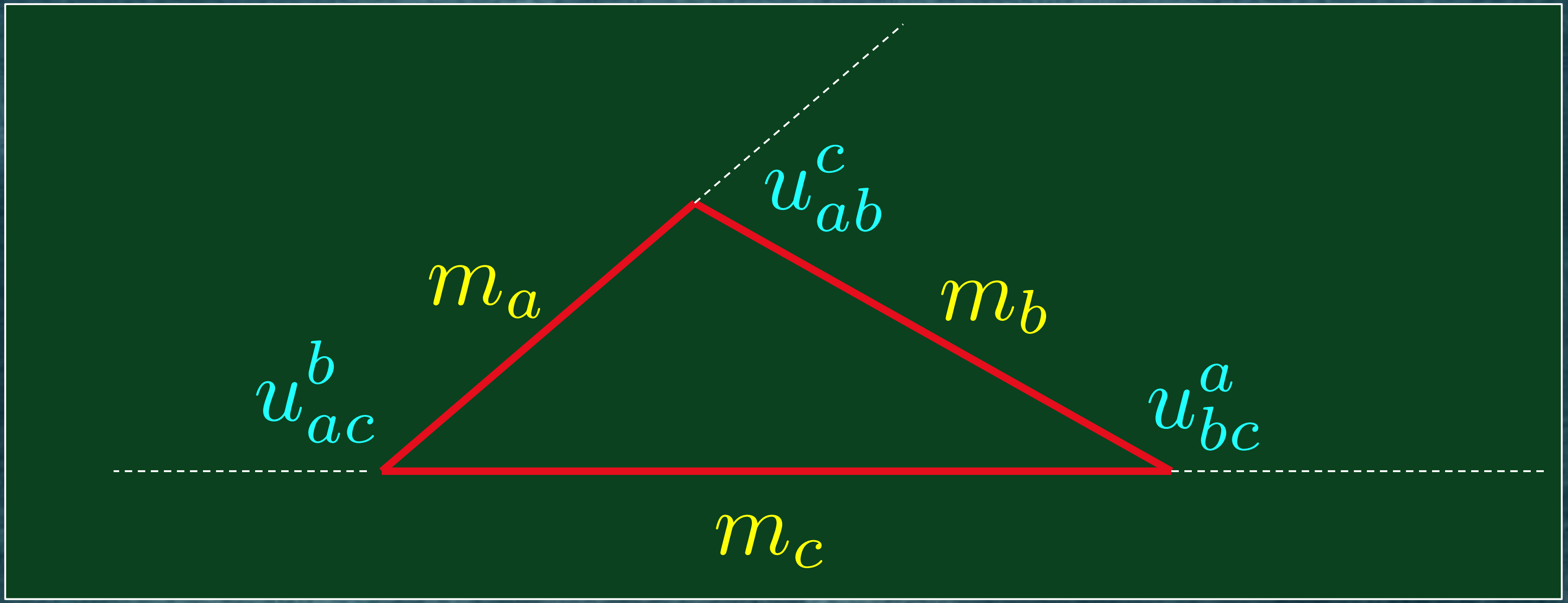


$$S_{ab}(\beta) \simeq i \frac{(\Gamma^c_{ab})^2}{\beta - i u^c_{ab}} \Rightarrow i \frac{(\Gamma^c_{ab})^2}{s - m_c^2}$$

Mandelstam variable and triangle of the masses

$$s(\beta) = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2 m_a m_b \cosh \beta$$

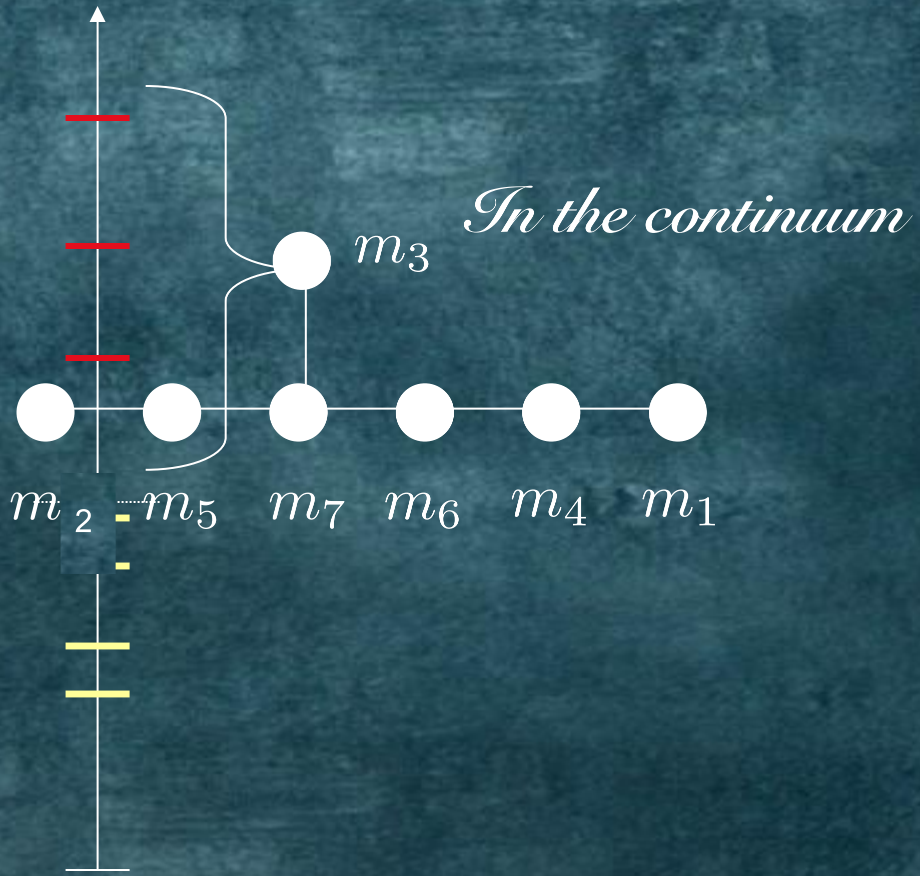
$$s(i u_{ab}^c) = m_a^2 + m_b^2 + 2 m_a m_b \cos u_{ab}^c$$



$$u_{ab}^c + u_{bc}^a + u_{ac}^b = 2\pi$$

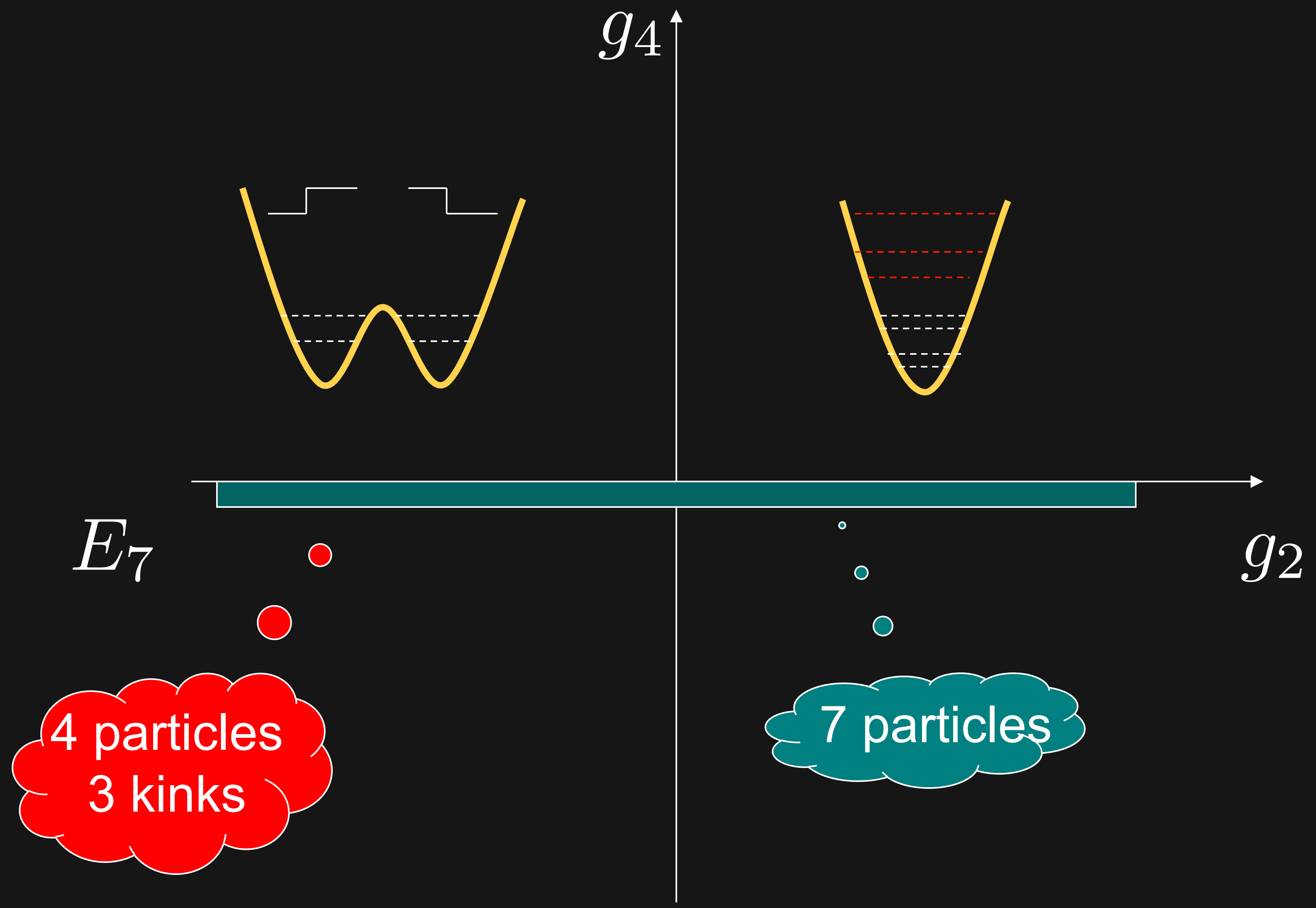
Mass spectrum of the Tricritical Ising along the thermal axis

$$m_1 = C\tau^{5/9} \equiv 1$$
$$m_2 = 2m_1 \cos(5\pi/18) = 1.285...$$
$$m_3 = 2m_1 \cos(\pi/9) = 1.879...$$
$$m_4 = 2m_2 \cos(\pi/18) = 1.969..$$
$$m_5 = 2m_4 \cos(\pi/9) = 2.532..$$
$$m_6 = 2m_3 \cos(2\pi/9) = 2.879...$$
$$m_7 = 2m_3 \cos(\pi/18) = 3.702..$$



Structure of bound states

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_1	A_2 A_4	A_1 A_3	A_2 A_4 A_5	A_1 A_3 A_6	A_3 A_6	A_4 A_5 A_7	A_6
A_2		A_2 A_4 A_5	A_1 A_3 A_6	A_2 A_5	A_2 A_4 A_7	A_3	A_5 A_7
A_3			A_2 A_7	A_1	A_1 A_6	A_2 A_5 A_7	A_1 A_6
A_4				A_4 A_5 A_7	A_2 A_4 A_7	A_1 A_6	A_4 A_5
A_5					A_5	A_1 A_3	A_2 A_4 A_7
A_6						A_4 A_7	A_1 A_3 A_6
A_7							A_2 A_5 A_7


g_4
g_2
E_7
4 particles
3 kinks
7 particles

Exact mass spectrum

Mass	Value	Parity(ht)	Low temp
$m_1 = M$	1	odd	kink
$m_2 = 2M\cos\frac{5\pi}{18}$	1.285...	even	particle
$m_3 = 2M\cos\frac{\pi}{9}$	1.879...	odd	kink
$m_4 = 2M\cos\frac{\pi}{18}$	1.969...	even	particle
$m_5 = 2m_4\cos\frac{\pi}{9}$	2.532...	even	particle
$m_6 = 2m_3\cos\frac{2\pi}{9}$	2.879...	odd	kink
$m_7 = 2m_3\cos\frac{\pi}{18}$	3.702...	even	particle

An efficient numerical method: TCSA approach

$$H = H_0 + \lambda \int_0^R dx\, \phi(x) =$$
$$= H_{CFT} + V$$

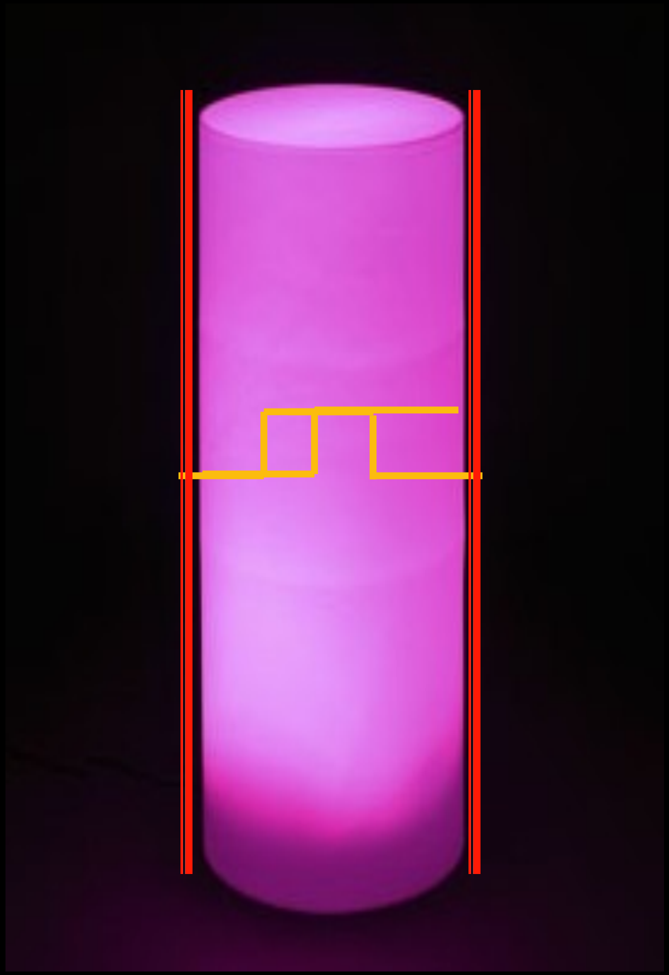
Matrix elements on the conformal states

$$< n \,|\, H_{CFT} \,|\, m > \; = \frac{2\pi}{R}\left(\Delta_n + \bar{\Delta}_n - \frac{c}{12}\right) \delta_{n,m}$$

$$< n \,|\, V \,|\, m > \; = \lambda \left(\frac{2\pi}{R}\right)^{2\Delta_\varphi - 1} C^{\varphi}_{nm}$$

$$H = \frac{2\pi}{R}\begin{pmatrix} *\,00000000\ldots\ldots \\ 0\,*\,0000000\ldots\ldots \\ 00\,*\,000000\ldots\ldots \\ 0\,000\,*\,0000\ldots\ldots \\ 0\,0000\,*\,000\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix} + R^{1-2\Delta}\begin{pmatrix} ***********\ldots \\ ***********\ldots \\ ***********\ldots \\ ***********\ldots \\ ***********\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix}$$

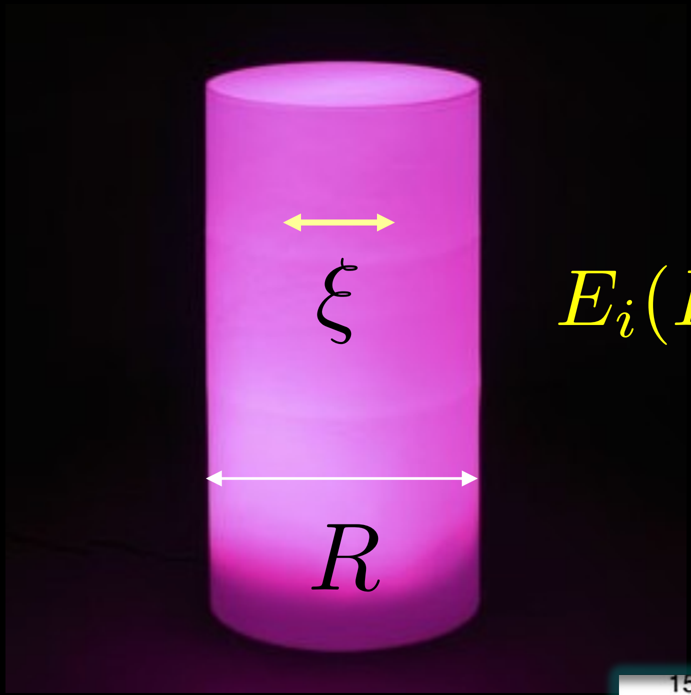
Role of the boundary conditions



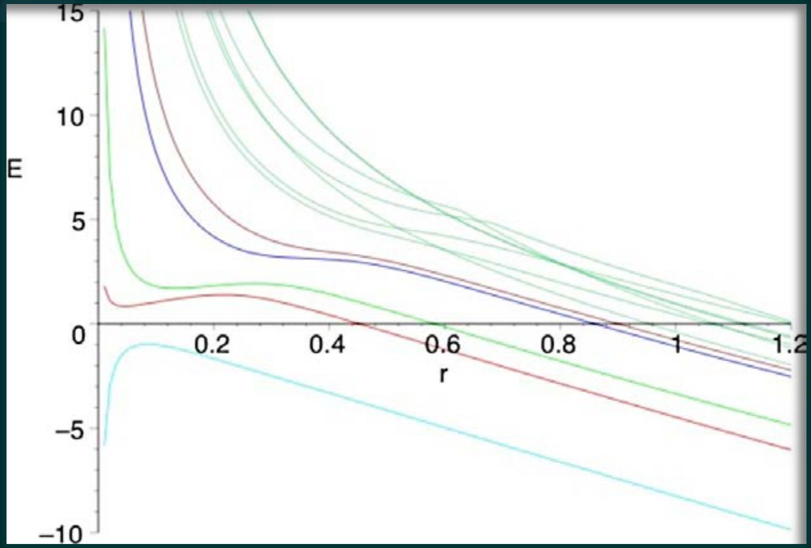
- Kink state exist only with anti-periodic or twisted b.c.

Only kink-antikink states are present with periodic b.c.

Behavior of the eigenvalues

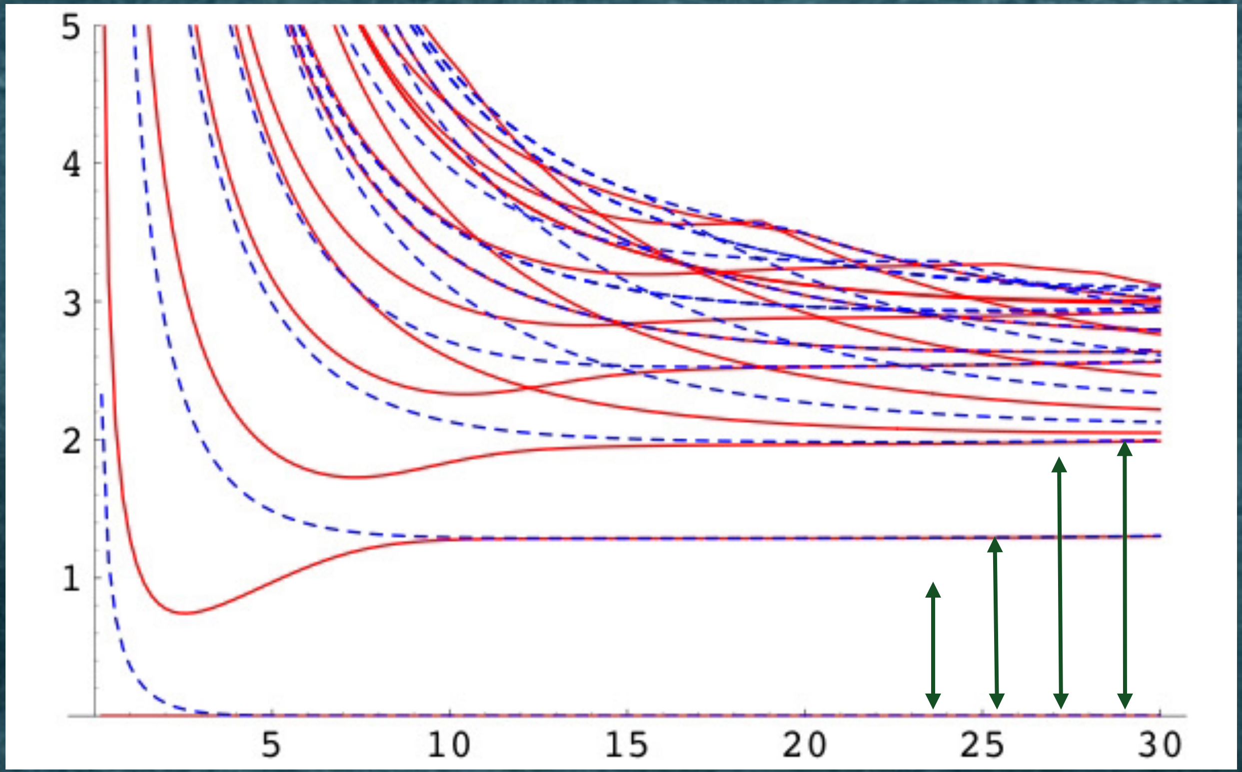


$$E_i(R) \simeq \begin{cases} \frac{2\pi}{R}\left(\Delta_i + \overline{\Delta}_i - \frac{c}{12}\right) & , \quad R \ll \xi \\ \frac{\epsilon_o}{\xi^2} R + \sum_i m_i & , \quad R \gg \xi \end{cases}$$



E_7 Numerical spectrum

$E_i - E_0$



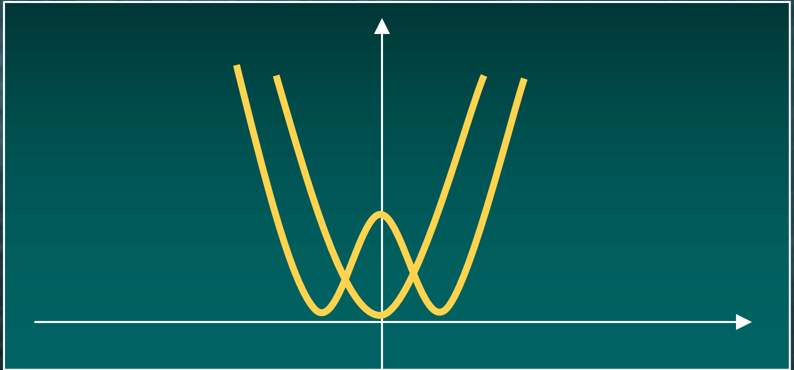
$m_4 = 1.96..$

$m_3 = 1.87..$

$m_2 = 1.28..$

$m_1 = 1$

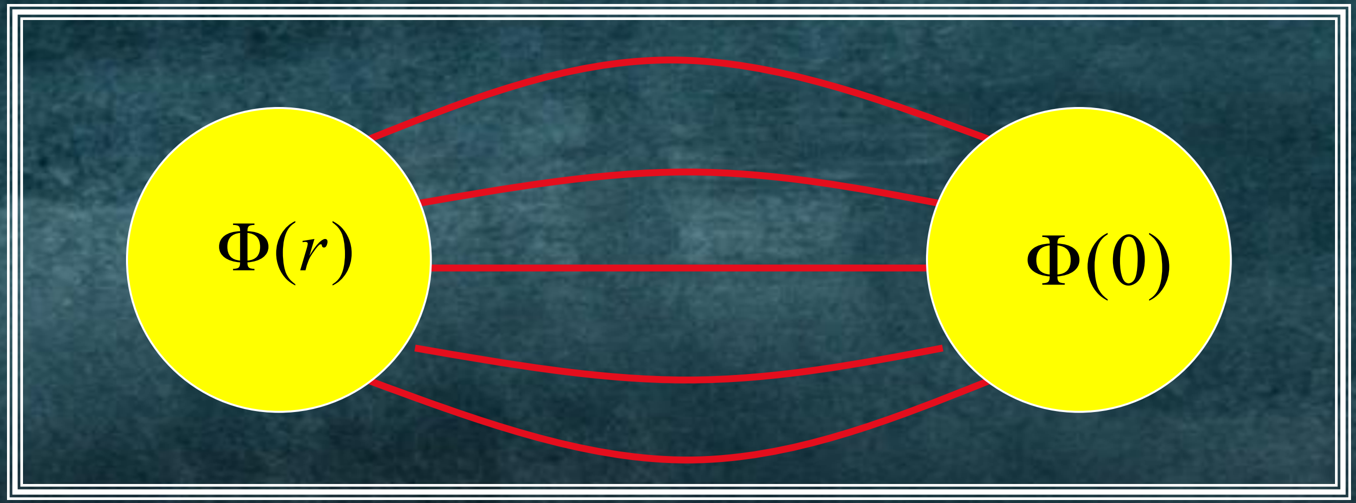
R



Correlators and their spectral representation

$$\langle 0|\Phi(r)\Phi(0)|0\rangle \;=\; \sum_n \langle 0|\Phi(r)|n\rangle\,\langle n|\Phi(0)|0\rangle$$

$$= \sum_{n=0}^{\infty} \int \prod_{j=1}^{n} \frac{d\beta_i}{2\pi}\, |\langle 0|\Phi(0)|A_1 \cdots A_n\rangle|^2 \;\times \exp\left(-r \sum_{i=1}^{n} m_i \cosh\beta_i\right)$$



Dynamic Structure Factors

$$D(\omega, q=0) \; = \int dx d\tau e^{i\omega\tau - iqx} \langle 0|\Phi(x,t)\Phi(0,0)|0\rangle \, |_{q=0}$$

$$= \; (2\pi)^2 \sum_{n=0}^{\infty} \int \prod_{i}^{n} \frac{d\beta_i}{2\pi} \left| \langle 0|\Phi(0)|A_1 \cdots A_n\rangle \right|^2 \delta(\omega - E_n)\, \delta(P_n)$$

All these quantities depend on the matrix elements on asymptotic particles, i.e. the so-called **Form Factors**

Duality and Form Factors in the Thermally Deformed Two-Dimensional Tricritical Ising Model

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2 Condensed Matter and Materials Physics Division, Brookhaven National Laboratory, Upton, NY 11973-5000, USA
3 BME-MTA Statistical Field Theory 'Lendület' Research Group, Department of Theoretical Physics, Budapest University of Technology and Economics 1111 Budapest, Budafoki út 8, Hungary
4 SISSA and INFN, Sezione di Trieste, via Bonomea 265, I-34136, Trieste, Italy
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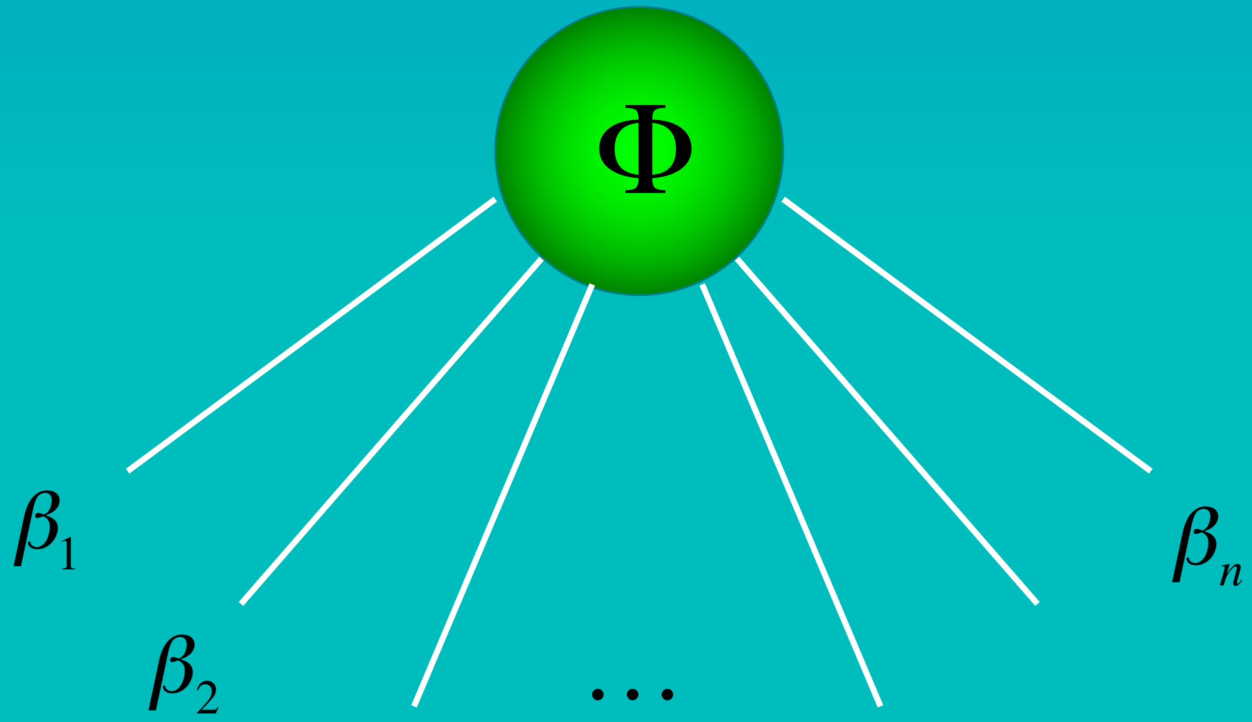
September 22, 2021

Abstract

The thermal deformation of the critical point action of the 2D tricritical Ising model gives rise to an exact scattering theory with seven massive excitations based on the exceptional E_7 Lie algebra. The high and low temperature phases of this model are related by duality. This duality guarantees that the leading and sub-leading magnetisation operators, $\sigma(x)$ and $\sigma'(x)$, in either phase are accompanied by associated disorder operators, $\mu(x)$ and $\mu'(x)$. Working specifically in the high temperature phase, we write down the sets of bootstrap equations for these four operators. For $\sigma(x)$ and $\sigma'(x)$, the equations are identical in form and are parameterised by the values of the one-particle form factors of the two lightest $\mathbb{Z}_2$ odd particles. Similarly, the equations for $\mu(x)$ and $\mu'(x)$ have identical form and are parameterised by two elementary form factors. Using the clustering property, we show that these four sets of solutions are eventually not independent; instead, the parameters of the solutions for $\sigma(x)/\sigma'(x)$ are fixed in terms of those for $\mu(x)/\mu'(x)$. We use the truncated conformal space approach to confirm numerically the derived expressions of the matrix elements as well as the validity of the Δ-sum rule as applied to the off-critical correlators. We employ the derived form factors of the order and disorder operators to compute the exact dynamical structure factors of the theory, a set of quantities with a rich spectroscopy which may be directly tested in future inelastic neutron or Raman scattering experiments.

Form Factors

$$F^{\Phi}_{i_1,i_2,\ldots,i_n}(\beta_1,\beta_2,\ldots,\beta_n) \equiv \langle 0 | \Phi(0) | A_{i_1}(\beta_1) A_{i_2}(\beta_2) \cdots A_{i_n}(\beta_n) \rangle$$



Watson's equations

$$\sum_{m=0}^{\infty} |\beta_1 \cdots \beta_m>_{out\ out}<\beta_1 \cdots \beta_m| = 1$$

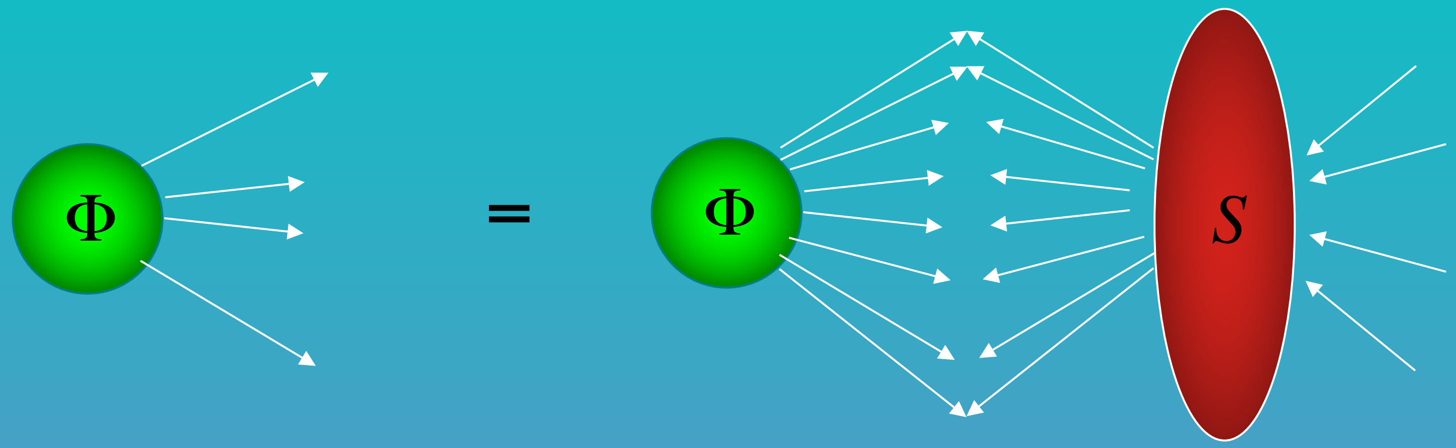
$$<0|\Phi(0)|\beta_1 \cdots \beta_n>_{in} = \sum_{m=0}^{\infty} <0|\Phi(0)|\beta_1 \cdots \beta_m>_{out\ out}<\beta_1 \cdots \beta_m|\beta_1 \cdots \beta_n>_{in}$$

$$= \sum_{m=0}^{\infty} <0|\Phi(0)|\beta_1 \cdots \beta_m>_{out} S_{n\to m}$$

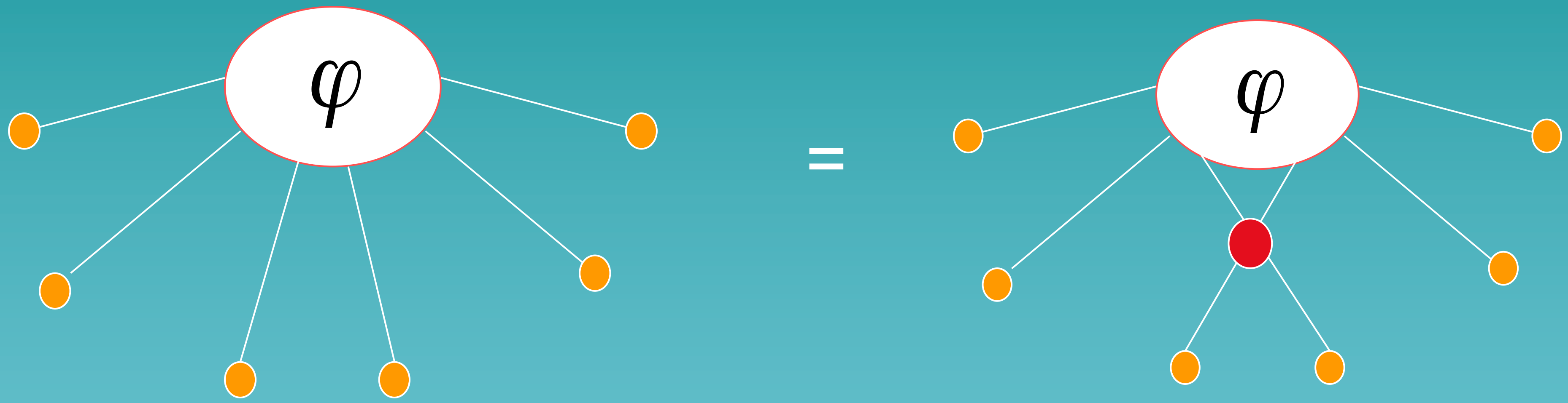
$$= <0|\Phi(0)|\beta_1 \cdots \beta_n>_{out} S_{n\to n}$$

The computation reduces to solve a Riemann-Hilbert problem !

But, in integrable theory the scattering is elastic !



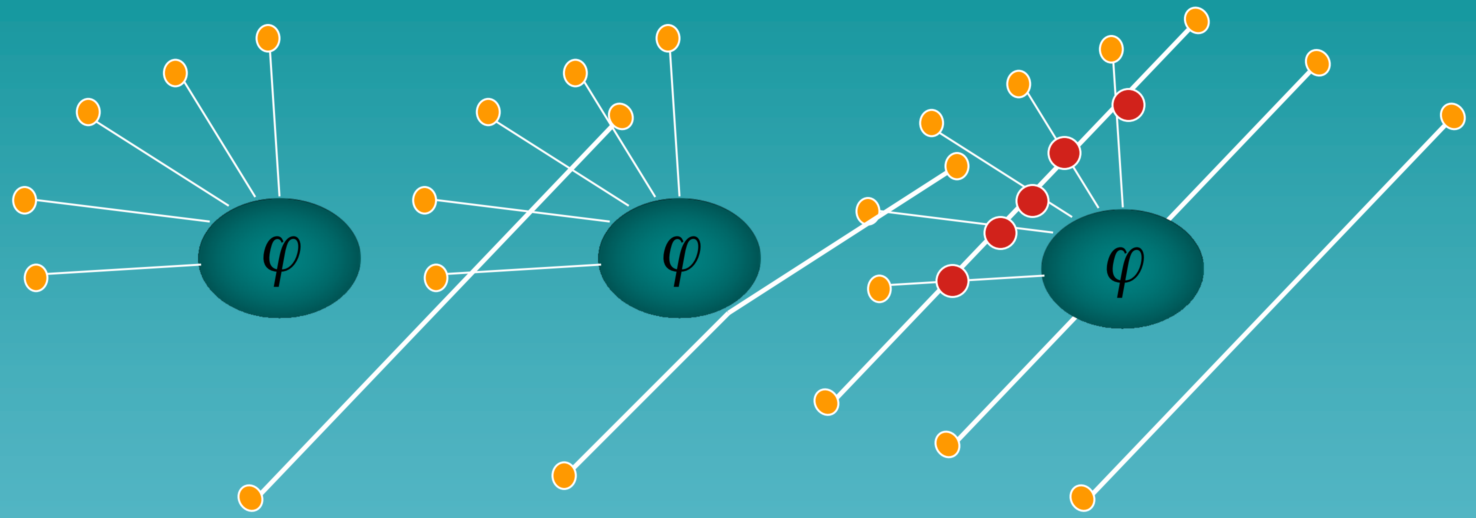
Monodromy properties



$$F(\beta_1,\ldots\beta_i,\beta_{i+1,}\ldots\beta_n) = S(\beta_i - \beta_{i+1})\; F(\ldots\beta_{i+1},\beta_i\ldots)$$

$$F(\beta_1 + 2\pi i,\ldots\beta_i,\beta_{i+1,}\ldots\beta_n) = F(\beta_2\ldots\beta_{i+1},\beta_i\ldots\beta_1)$$

Kinematic Recursive Equations

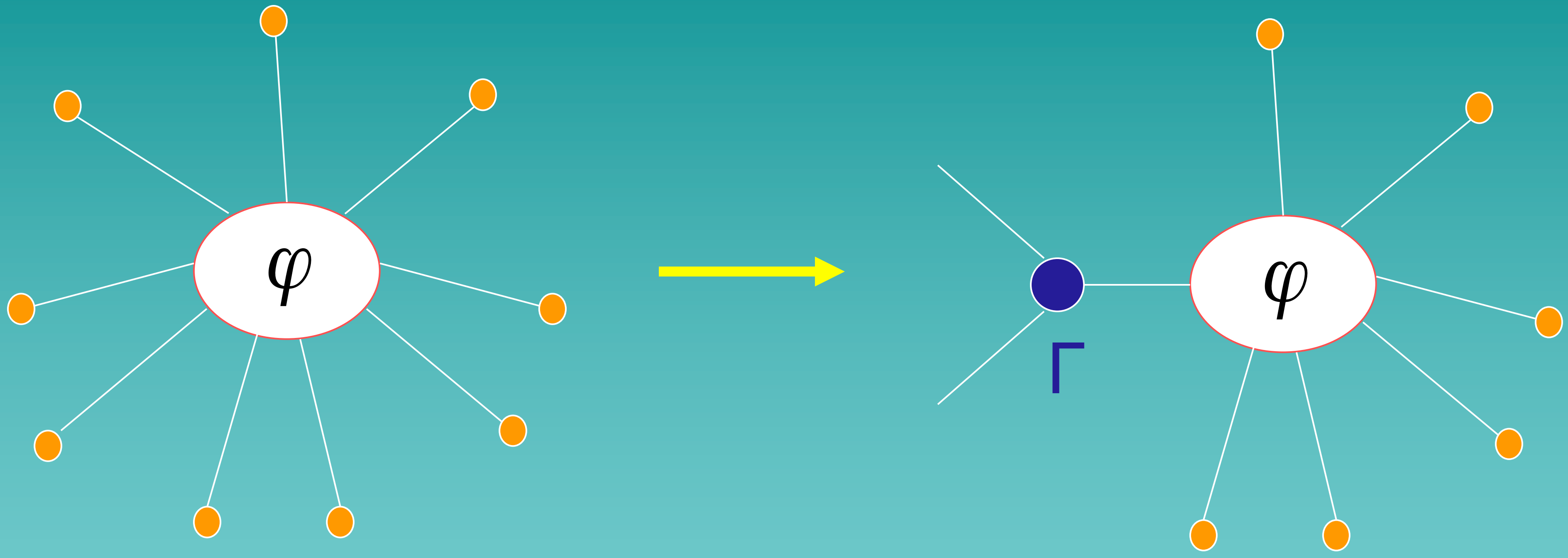


$$-i \lim_{\beta \to i\beta'} (\beta - i\beta')\, F_n(\beta' + i\pi, \beta, \ldots, \beta_n) =$$

$$\left[1 - e^{2\pi i \gamma} \prod_{i=1}^{n} S(\beta - \beta_i) \right] F_{n-2}(\beta_1, \ldots, \beta_n)$$

$$F_n \longrightarrow F_{n+2}$$

Bound State Recursive Equations



$$-i \lim_{\beta_{ab} \to i u_{ab}^{c}} (\beta_{ab} - i u_{ab}^{c})\, F_n(\beta_a, \beta_b, \ldots, \beta_n) = \Gamma_{ab}^{c}\, F(\beta_c, \beta_3, \ldots, \beta_n)$$

$$F_n \rightarrow F_{n-1}$$

Important observation

(Cardy-Mussardo, Nucl.Phys. B340)

All equations we employed never refer to the operator of which we are computing the Form Factors !

This means that, finding all possible solutions of the FF and classifying them, we can obtain the

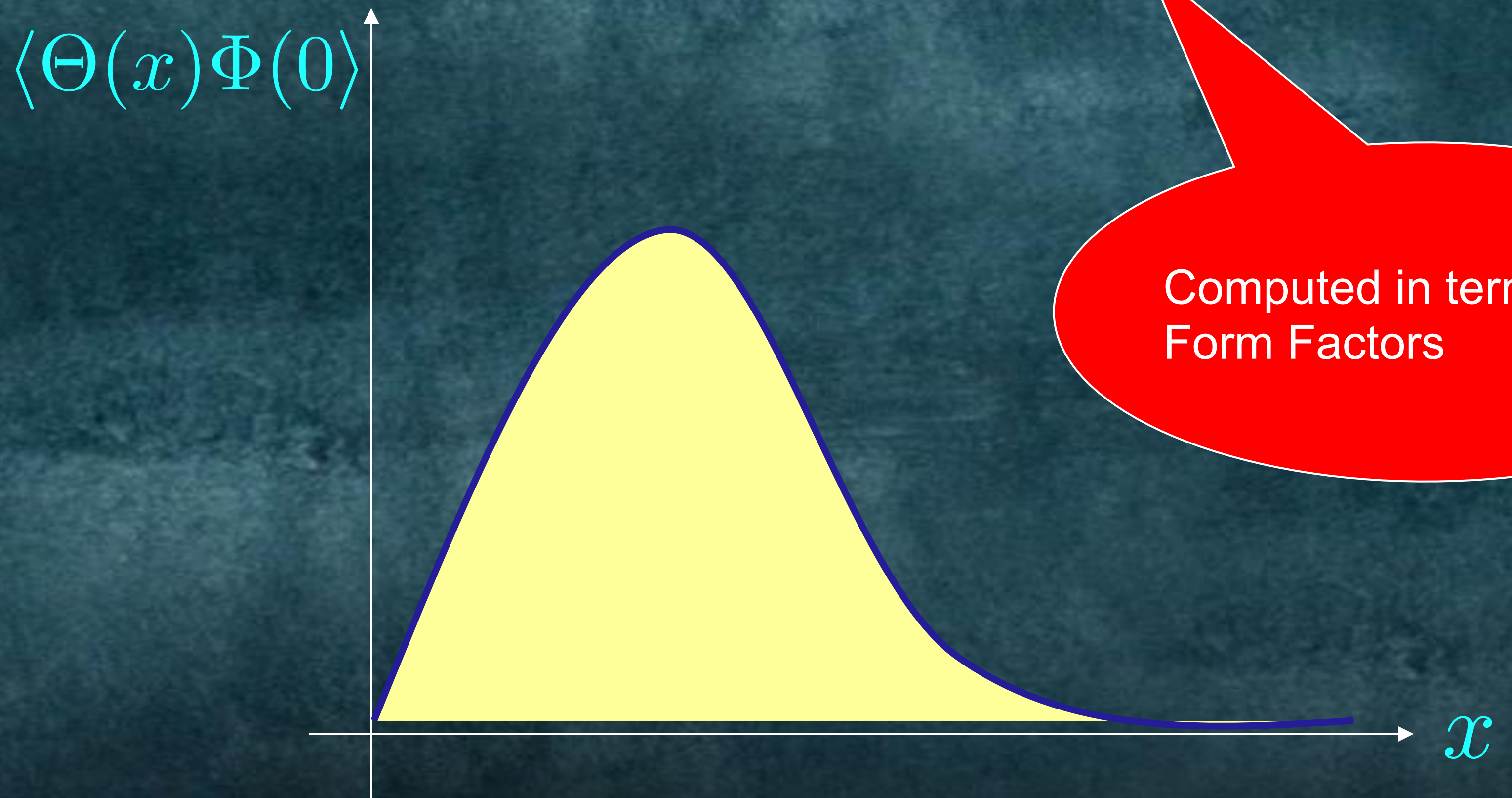
Operator Content

For the Tricritical Ising Model…

- 2 Z_2 odd local solutions — order parameters → $\sigma(x)$, $\tilde{\sigma}(x)$
- 2 Z_2 even non-local solutions — disorder parameters → $\mu(x)$, $\tilde{\mu}(x)$
- 2 Z_2 even local solutions — energy density → $\epsilon(x)$; vacancy density → $t(x)$

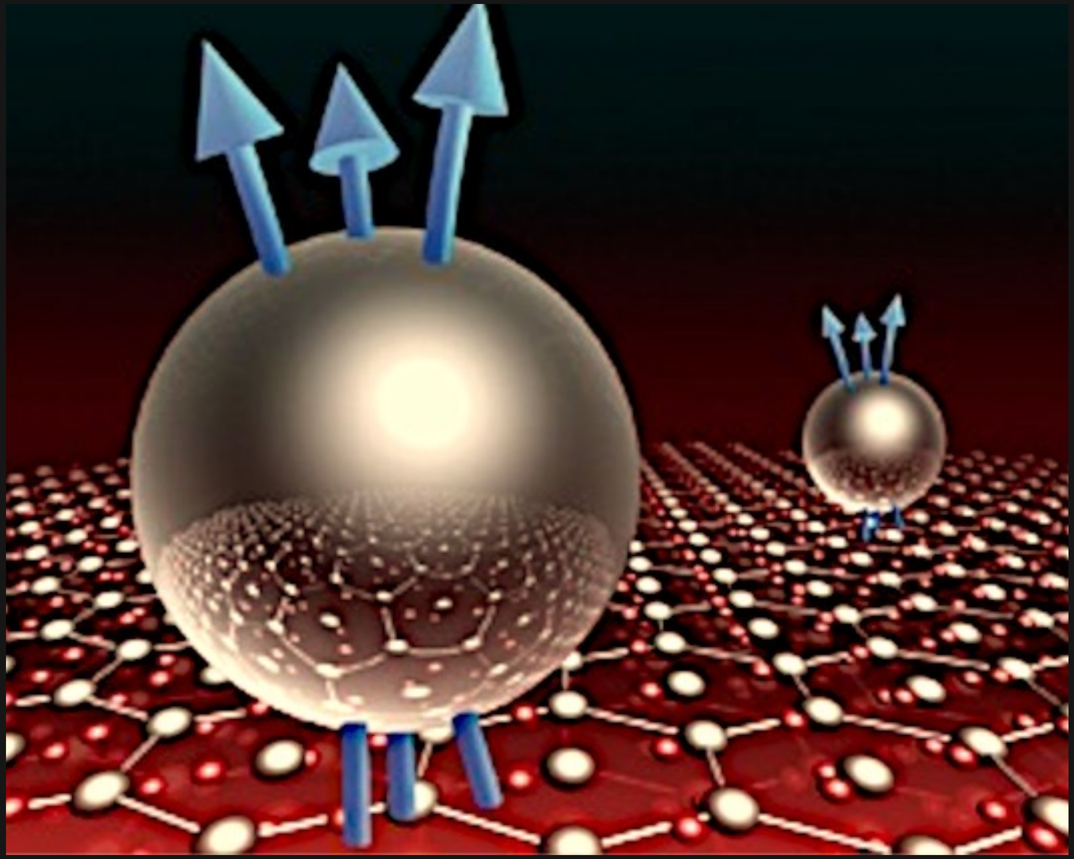
Delta sum rule

$$\Delta = -\frac{1}{4\pi\langle\Phi\rangle}\int dx\langle\Theta(x)\Phi(0)\rangle$$



Computed in terms of the Form Factors

Lattice variable vs field operators



$$S_{ex}(x) \simeq A\,\sigma_{th}(x) + B\,\tilde{\sigma}_{th}(x) + \mathrm{irr}$$

Exact predictions for structure functions

$$\langle S(x)S(0)\rangle_{ex} = \langle (A\sigma(x) + B\tilde{\sigma}(x))\,(A\sigma(0) + B\tilde{\sigma}(0))\rangle$$

$$= A^2\,\langle\sigma(x)\sigma(0)\rangle_{th} + B^2\,\langle\tilde{\sigma}(x)\tilde{\sigma}(0)\rangle_{th} + 2AB\,\langle\sigma(x)\tilde{\sigma}(0)\rangle_{th}$$

Exact predictions for structure functions

$$D_{ex}^{SS}(\omega) = A^2 D_{th}^{\sigma\sigma}(\omega) + B^2 D_{th}^{\tilde{\sigma}\tilde{\sigma}}(\omega) + 2AB\, D_{th}^{\sigma\tilde{\sigma}}(\omega)$$

A and B constants which can be easily fixed by the resonance peaks!

Exact predictions for structure functions

- Z_2 odd (high-temperature)

$$D^{\sigma\sigma}(\omega) = \int e^{i\omega t} \langle \sigma(t)\sigma(0) \rangle$$

$$D^{\tilde{\sigma}\tilde{\sigma}}(\omega) = \int e^{i\omega t} \langle \tilde{\sigma}(t)\tilde{\sigma}(0) \rangle$$

$$D^{\sigma\tilde{\sigma}}(\omega) = \int e^{i\omega t} \langle \sigma(t)\tilde{\sigma}(0) \rangle$$

- Z_2 even (low-temperature)

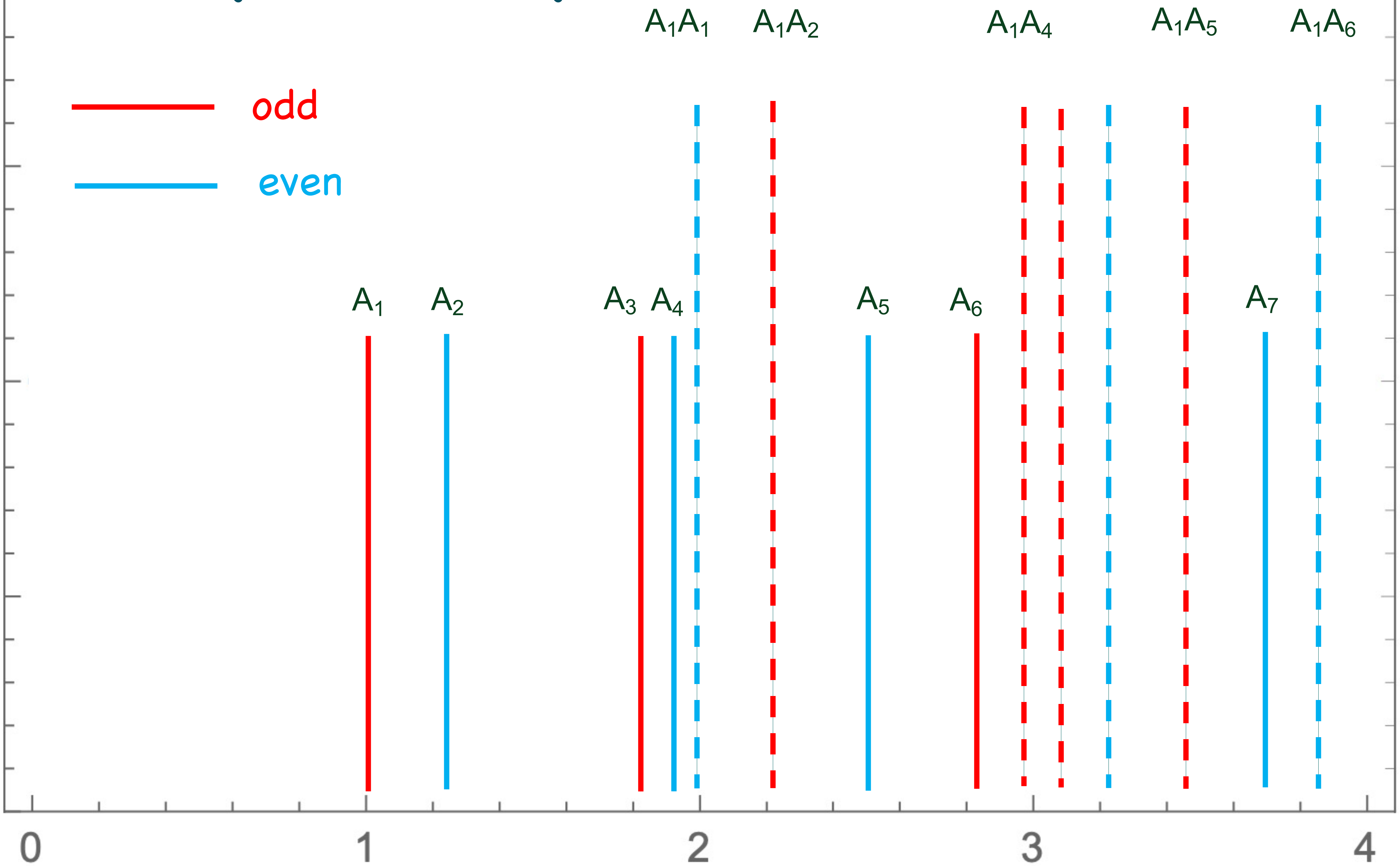
$$D^{\mu\mu}(\omega) = \int e^{i\omega t} \langle \mu(t)\mu(0) \rangle$$

$$D^{\tilde{\mu}\tilde{\mu}}(\omega) = \int e^{i\omega t} \langle \tilde{\mu}(t)\tilde{\mu}(0) \rangle$$

$$D^{\tilde{\mu}\mu}(\omega) = \int e^{i\omega t} \langle \tilde{\mu}(t)\mu(0) \rangle$$

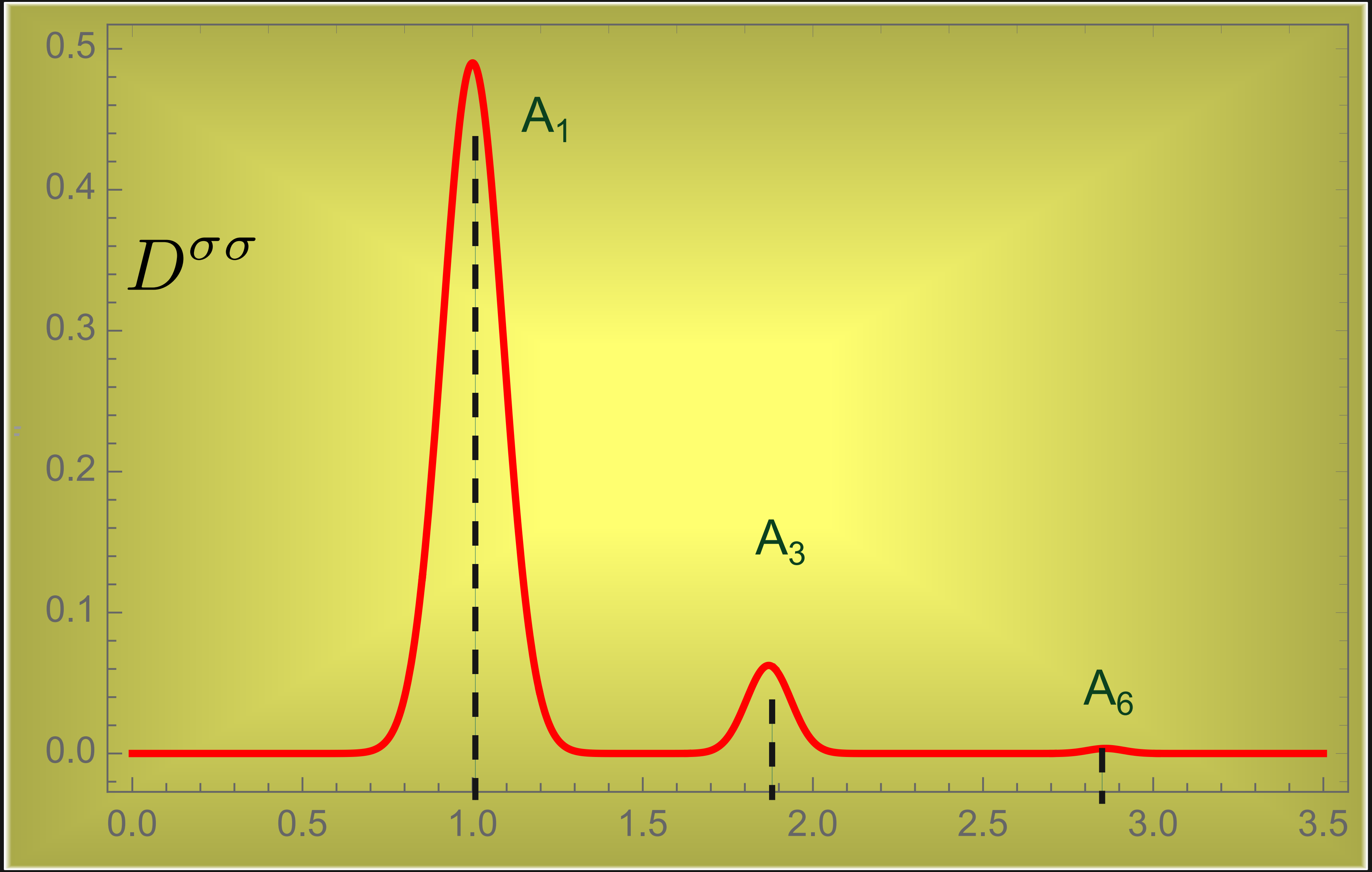
state	ω/m_1	parity	state	ω/m_1	parity
A_2	1.28558	even	A_1	1.00000	odd
A_4	1.96962	even	A_3	1.87939	odd
$A_1\ A_1$	≥ 2.00000	even	$A_1\ A_2$	≥ 2.28558	odd
$A_2\ A_2$	≥ 2.57115	even	A_6	≥ 2.87939	odd
$A_1\ A_3$	≥ 2.87939	even	$A_1\ A_4$	≥ 2.96952	odd
A_5	2.53209	even	$A_2\ A_3$	≥ 3.16496	odd
$A_2\ A_4$	≥ 3.25519	even	$A_1\ A_5$	≥ 3.53209	odd
A_7	3.70167	even	$A_3\ A_4$	≥ 3.84901	odd
$A_3\ A_3$	≥ 3.75877	even			
$A_2\ A_5$	≥ 3.81766	even			
$A_1\ A_6$	≥ 3.87939	even			

E_7 Spectroscopy



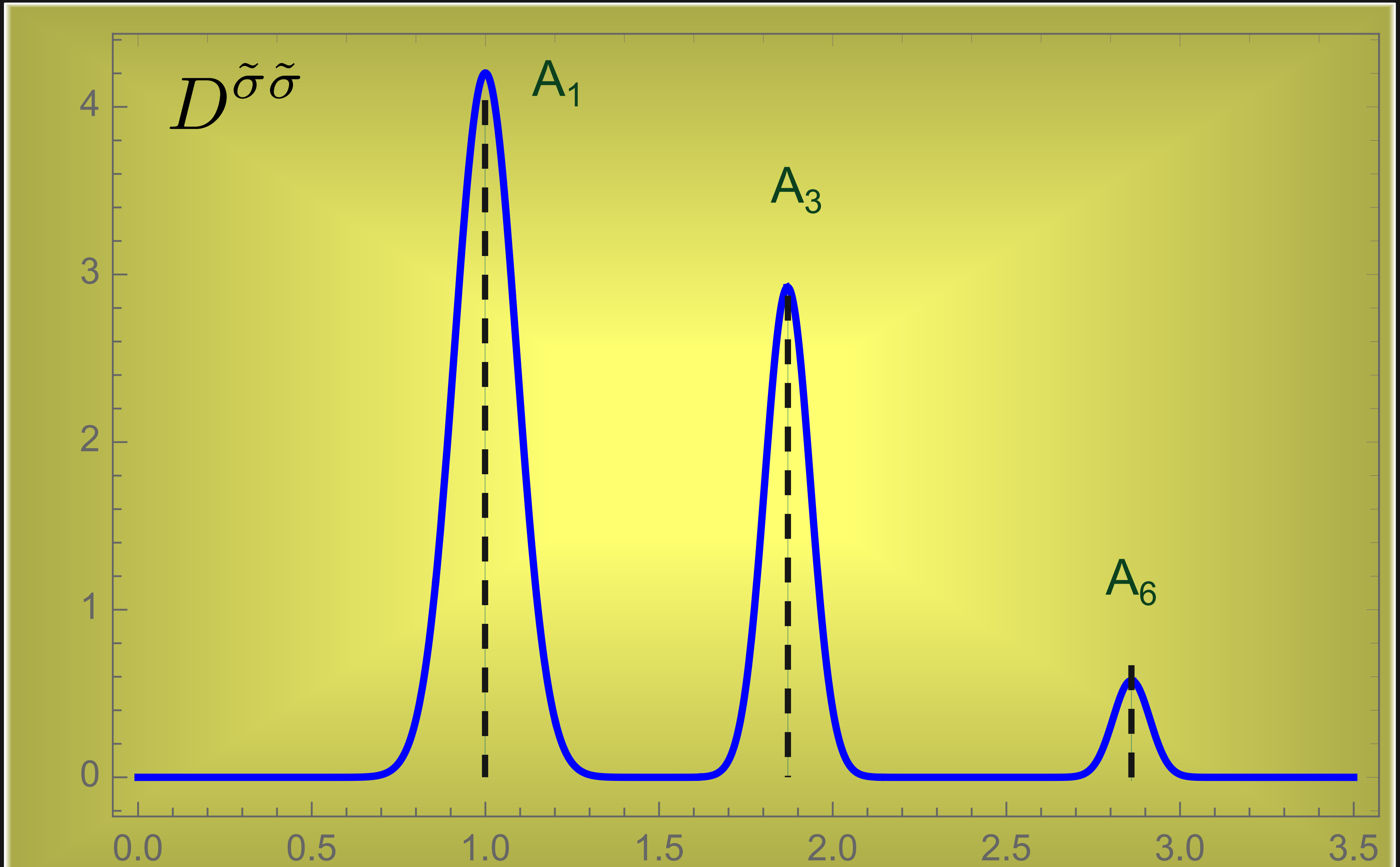
Order parameter structure functions (High Temperature)

One-particle contributions



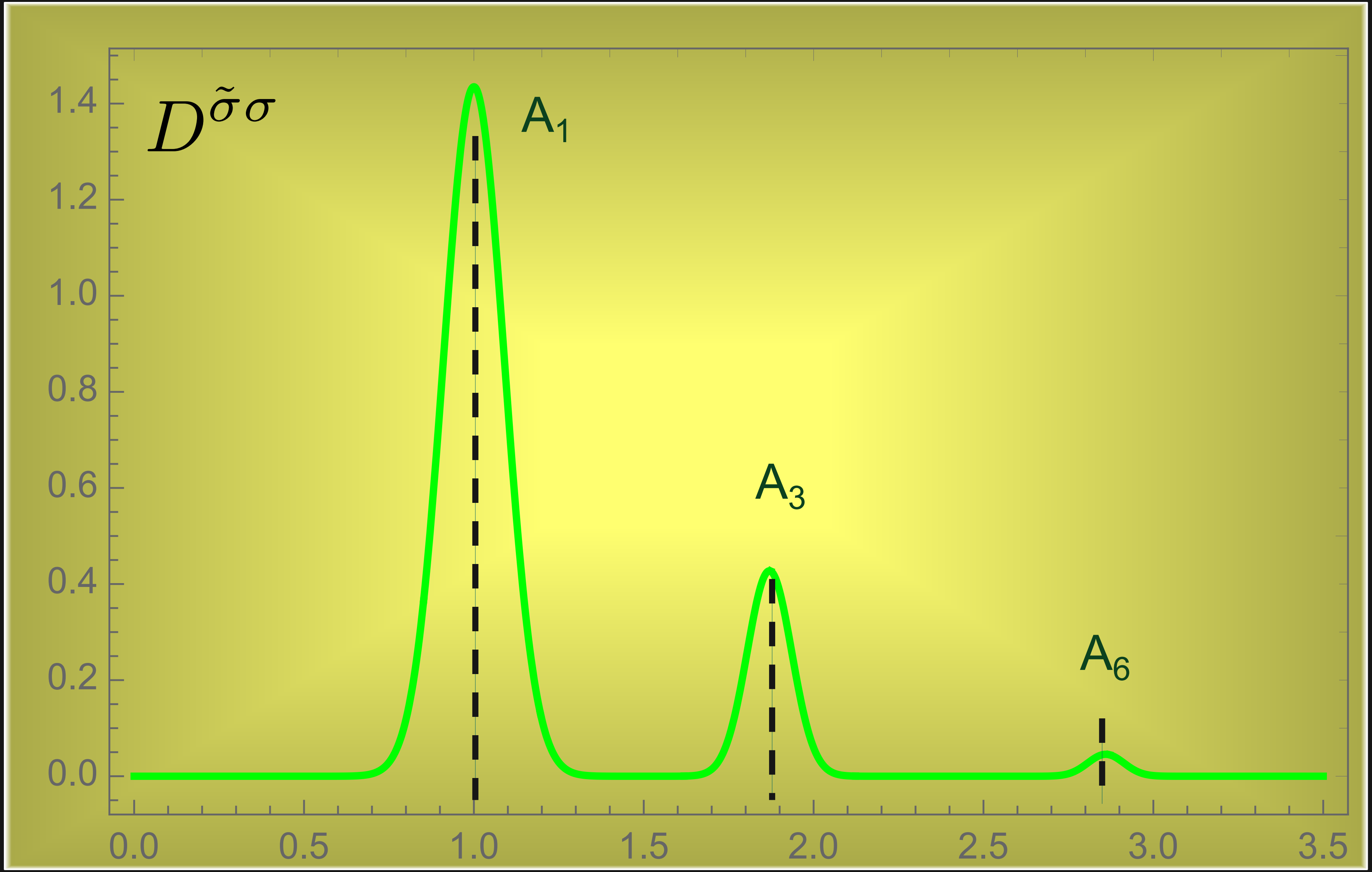
Order parameter structure functions (High Temperature)

One-particle contributions



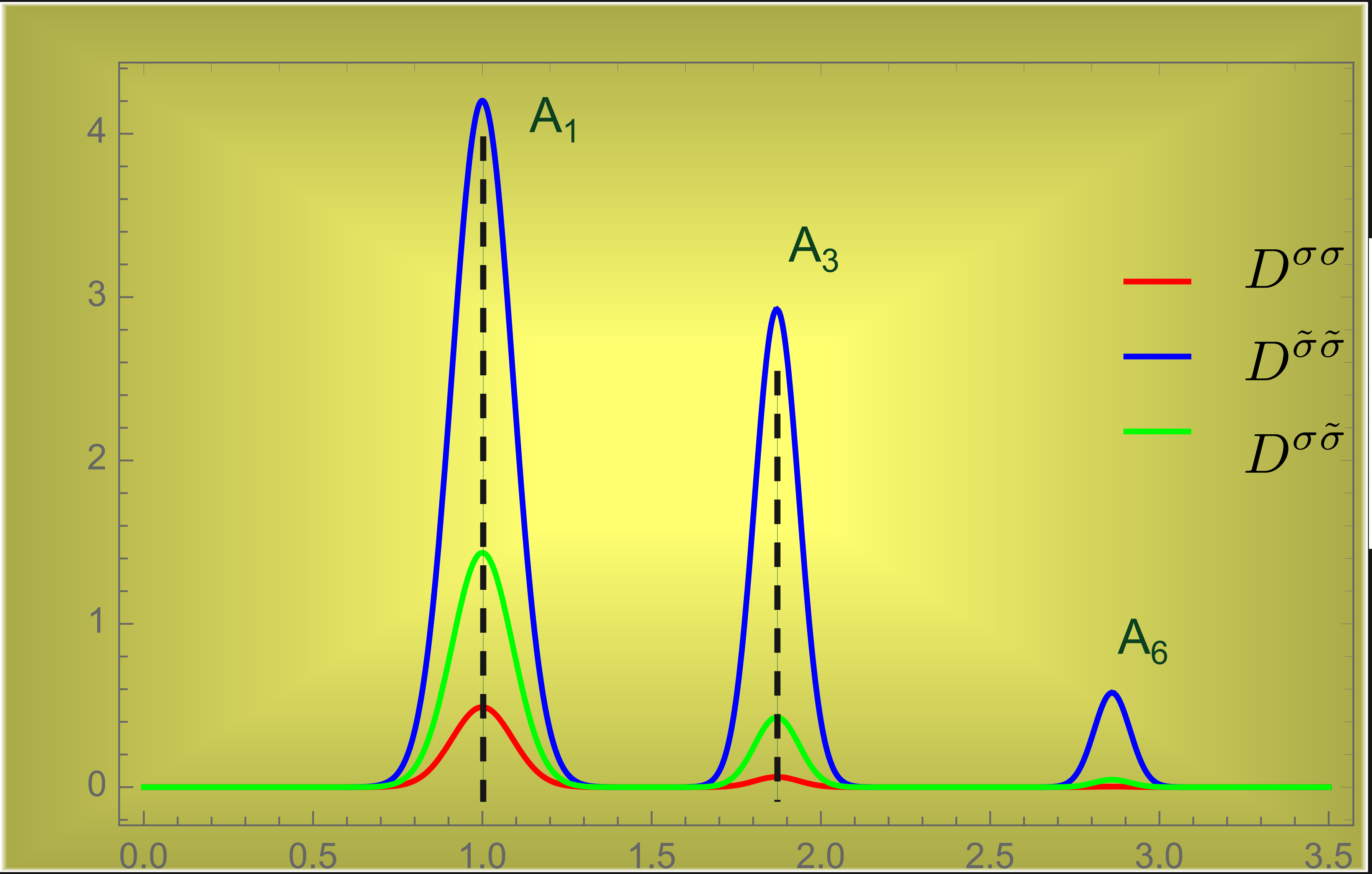
Order parameter structure functions (High Temperature)

One-particle contributions



Order parameter structure functions (High Temperature)

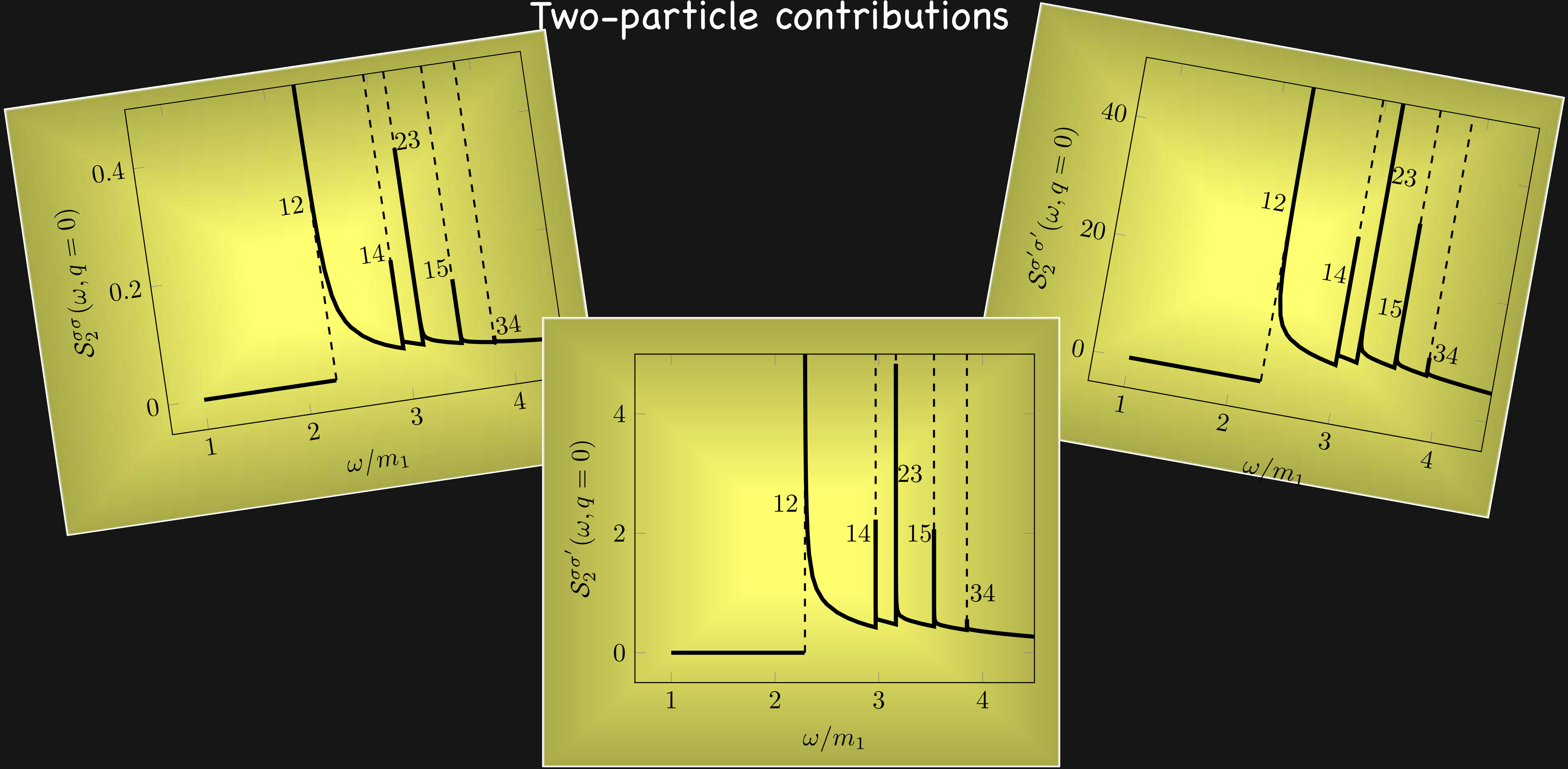
One-particle contributions



particle	$2\pi \frac{\lvert F_i^{\sigma}\rvert^2}{m_i}$	$2\pi \frac{\lvert F_i^{\sigma'}\rvert^2}{m_i}$	$2\pi \frac{F_i^{\sigma} F_i^{\sigma' *}}{m_i}$
1	3.17116	26.5578	9.1771
3	0.212838	9.82114	1.44579
6	0.0095296	1.29193	0.110957

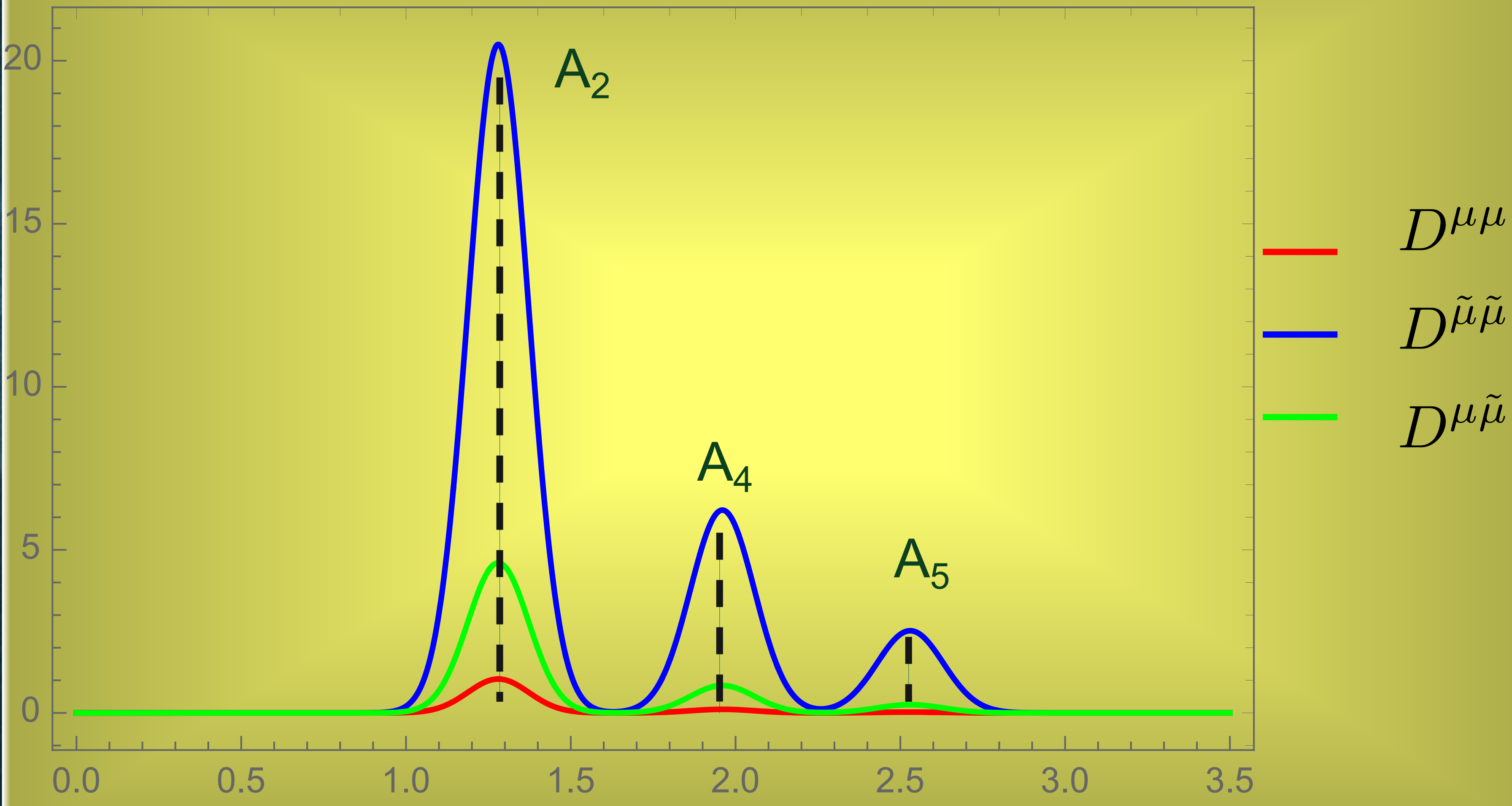
Order parameter structure functions (High Temperature)

Two-particle contributions



Disorder parameter structure functions (Low Temperature)

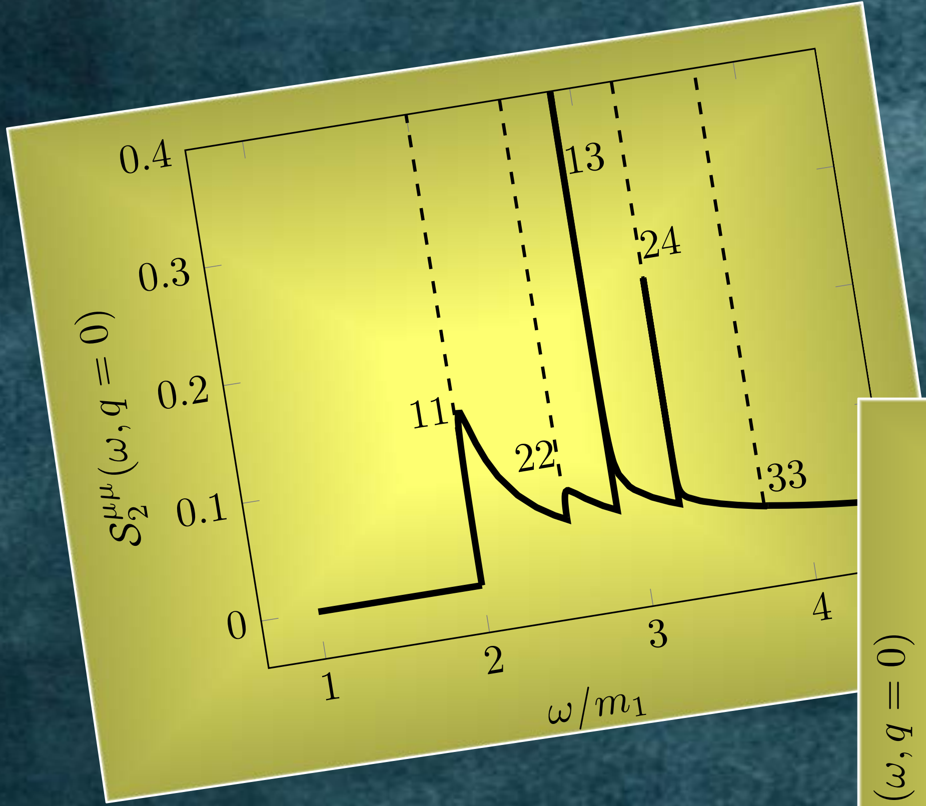
One-particle contributions

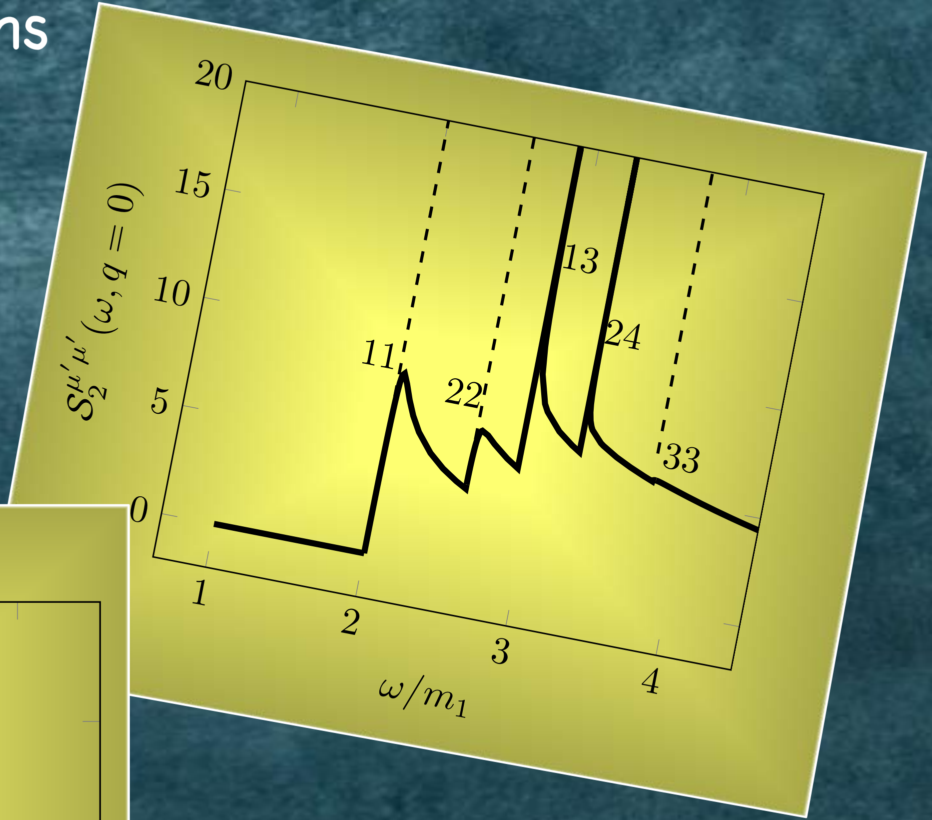


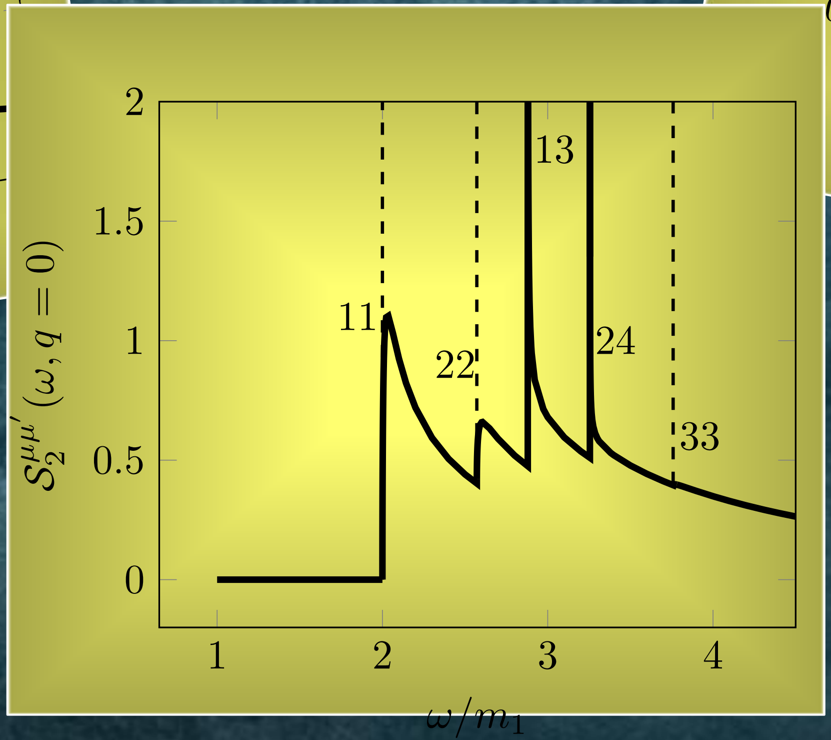
particle	$2\pi \frac{\lvert F_i^{\mu}\rvert^2}{m_i}$	$2\pi \frac{\lvert F_i^{\mu'}\rvert^2}{m_i}$	$2\pi \frac{F_i^{\mu} F_i^{\mu'*}}{m_i}$
2	1.04617	20.5658	4.63846
4	0.115915	6.22406	0.849389
5	0.0250625	2.52991	0.25805
7	0.000566863	0.146462	0.00911174

Disorder parameter structure functions (High Temperature)

Two-particle contributions







Overall Perspective and Challenges

- Physics of spin 1 displays an interesting web of remarkable symmetries
  (E_7, SUSY, duality, etc)
- The TIM is an ideal playground for this physics and a theoretical gem in itself
  (CFT, quantum integrability, etc.)
- Its self-duality is related to (interacting) fermions and their zero modes
- Detailed studies of the E7 structure of TIM
  (Elastic S-matrix, exact mass spectrum, Form Factors, etc.)
- Exact computation of structure functions, with their rich spectroscopy
- It would be extremely fascinating to realise such a class of universality
  experimentally!