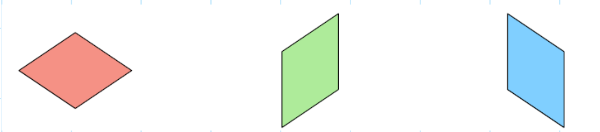
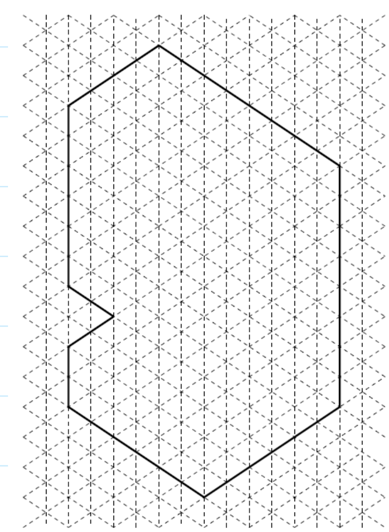
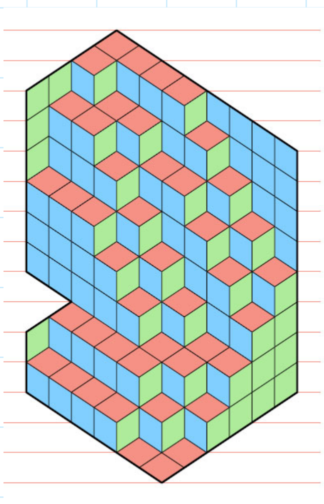
# Basic setup

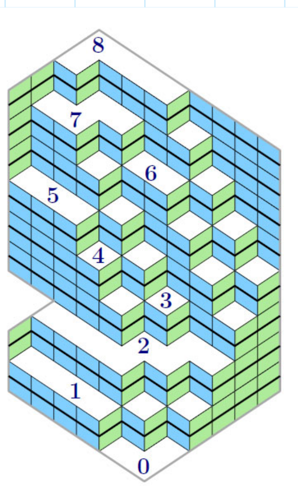


tile $\longrightarrow$



Sample tiling uniformly at random

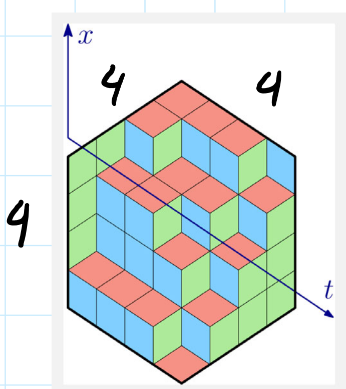


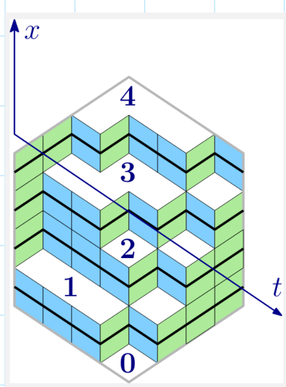



Height function

$H(T, X) = \#$paths below $(T, X)$

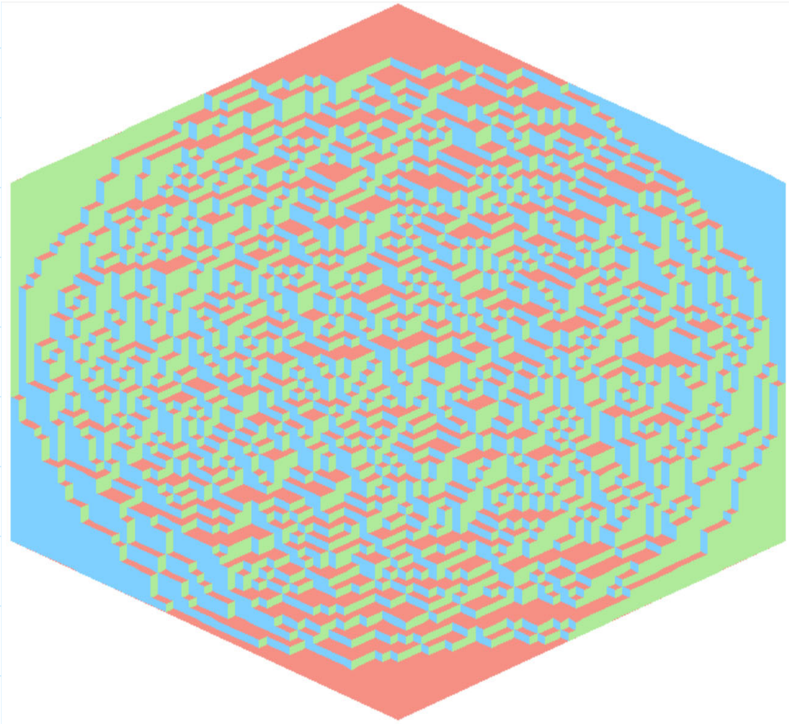
If you want it to be continuous $\rightarrow$ interpolate linearly



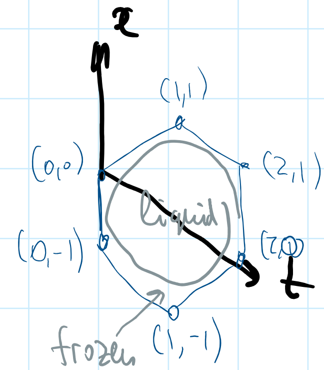




$N \times N \times N$ hexagon
$N \to \infty$

Uniformly r. tiling = ?



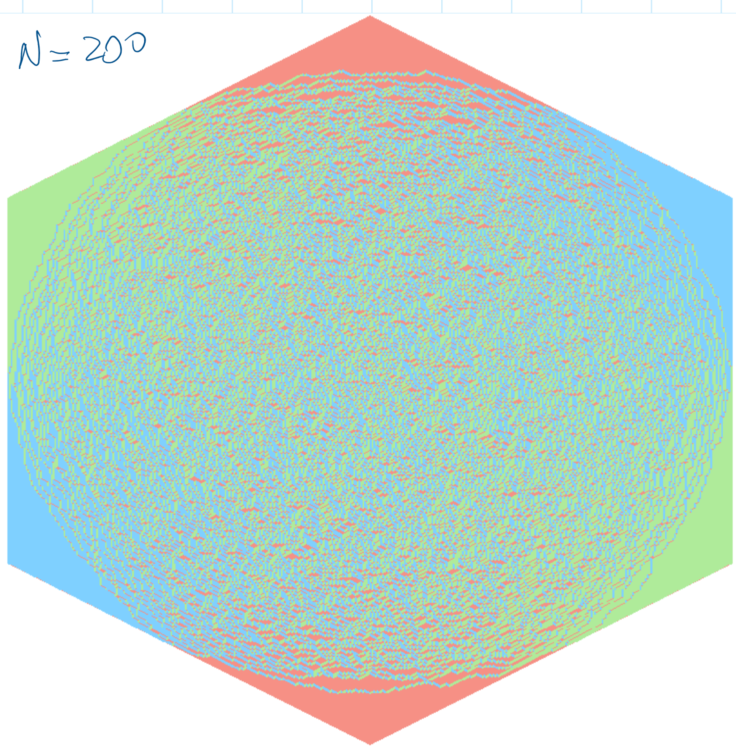
## N = 50




**Limit shape theorem:** $$\underbrace{\lim_{N\to\infty} \frac{H(tN, xN)}{N}}_{\text{random object}} = \overbrace{h(t,x)}^{\text{deterministic}}$$

height of uniformly random tiling





"frozen region"

$$\nabla h(t,x) = \begin{cases} (0,1) \\ (-1,1) \\ (0,0) \end{cases} \Bigg| \text{ outside inscribed circle}$$

$\in$ triangle of slopes inside the inscribed circle

liquid / rough region

New York Journal of Mathematics
New York J. Math. 4 (1998) 137–165.

**The Shape of a Typical Boxed Plane Partition**

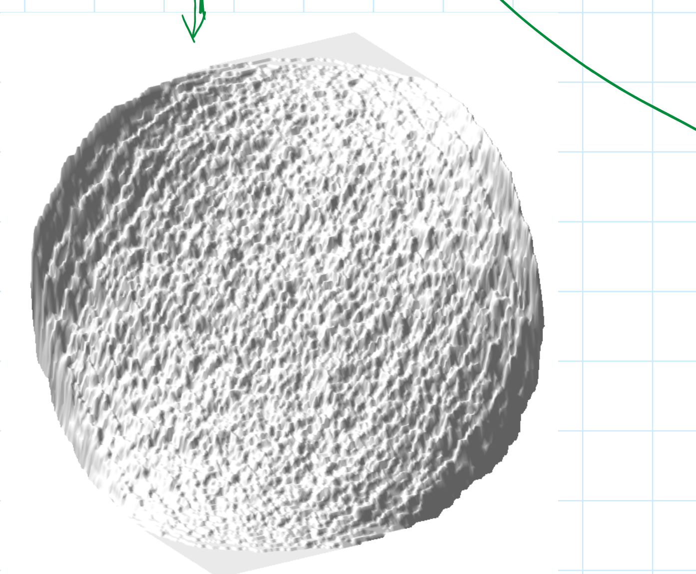
Henry Cohn, Michael Larsen, and James Propp

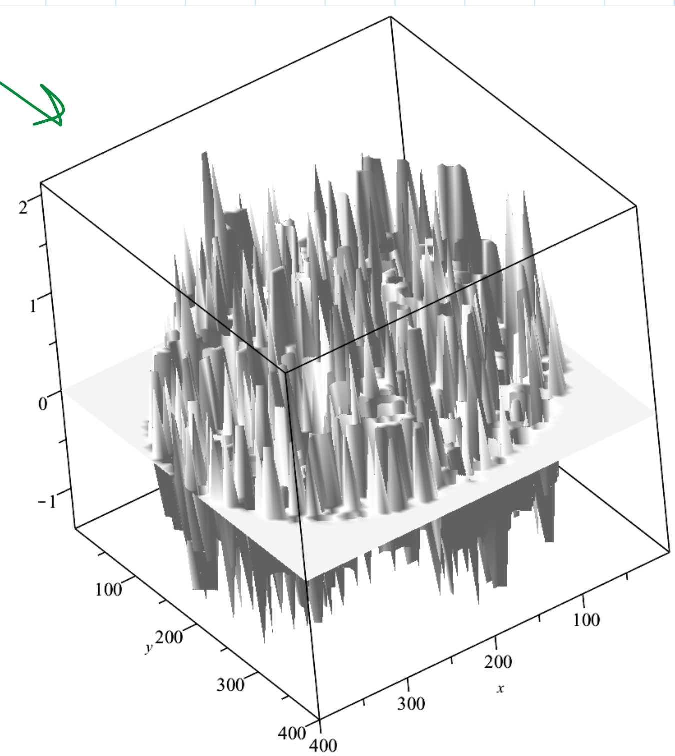
← Original proof of limit shape

## Global fluctuations theorem:

$\exists$ map $\Omega$ : interior of circle $\to$ upper halfplane $\mathbb{C}^+$ such that

$$\sqrt{\pi}\left(H(tN, xN) - \mathbb{E}\,H(tN, xN)\right) \xrightarrow{N\to\infty} \Omega\text{-pullback of the Gaussian Free Field (GFF) in } \mathbb{C}^+$$





**Def** GFF in $\mathbb{C}^+$ is a mean 0 (generalized) Gaussian random function

$$\mathrm{GFF}: \mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \to \mathbb{R}$$

such that
$$\mathbb{E}\,\mathrm{GFF}(z)\,\mathrm{GFF}(w) = -\frac{1}{2\pi} \ln \left| \frac{z-w}{z-\bar{w}} \right| \quad (*)$$

---

1) $\mathrm{GFF}(\text{real number}) = 0$ „Dirichlet boundary condition"

2) $\mathrm{GFF}(z)$ is not defined at a particular point $z$

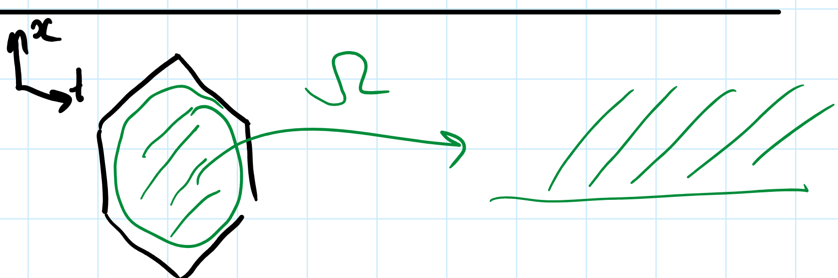
but $\int_{\mathbb{C}^+} \mathrm{GFF}(z)\, \mu(dz)$ is a bona fide Gaussian random variable for any sufficiently smooth measure $\mu(dz)$

3) $(*)$ is the Green function of the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in $\mathbb{C}^+$ with zero boundary conditions

Kernel of inverse operator

---

What is $\Omega$?




Consider a quadratic equation

$$(**) \quad (2-t)\,u^2 + (1-2x)\,u + (t-x) = 0$$

(**) $$(2-t)u^2 + (1-2x)u + (t-x) = 0$$

**Lemma:** It has two complex-conjugate roots for $(t,x)$ inside the inscribed circle and real roots otherwise.

**Proof** Complex roots iff $D<0$

$$D = (1-2x)^2 - 4(2-t)(t-x) = 4\left(x-\frac{t}{2}+\frac{1}{2}\right)^2 + 3(t-1)^2 - 3$$

$D<0$ inside the inscribed circle. $\boxtimes$

**Def** $\Omega(t,x)$ = solution to (**) in the upper half plane $\mathbb{C}^+$

**Theorem** For any "smooth" measure $\mu(dt,dx)$

$$\iint_{\text{hexagon}} \sqrt{\pi}\left(H(tN,xN) - \mathbb{E}H(tN,xN)\right)\mu(dt,dx)$$

$$\downarrow N\to\infty$$

$$\mathcal{N}(0,\sigma_\mu^2)$$ Gaussian

$$\sigma_\mu^2 = \iint_{(\text{hexagon})^2} -\frac{1}{2\pi}\ln\left|\frac{\Omega(t,x)-\Omega(t',x')}{\Omega(t,x)-\overline{\Omega}(t',x')}\right|\mu(dt,dx)\,\mu(dt',dx')$$

---

Remark: Individual $H(tN,xN) - \mathbb{E}H(tN,xN)$ grows as $\sqrt{\log N}$. Only after averaging we see as $\sqrt{\log N}$. Only after averaging we see something finite.

---

$\Omega(t,x)$ is complicated even for hexagon.

But there is a link to $h(t,x) = \lim_{N\to\infty} \frac{1}{N} H(tN, xN)$.

This is a link between LLN and CLT

deterministic limit — fluctuations

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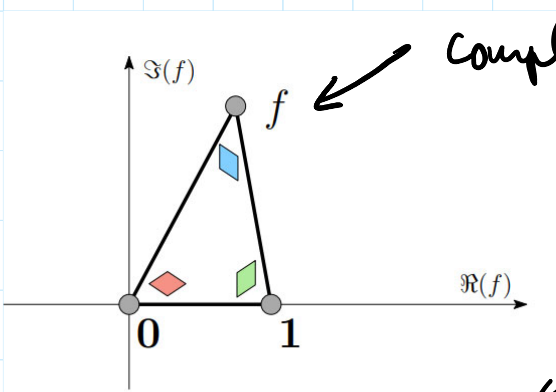
$(t,x)$ inside circle $\longrightarrow$ „complex slope" $f(t,x)$




Triangle $(0,1,f)$ has angles $(\pi p_{\lozenge}, \pi p_{\square}, \pi p_{\square})$.

complex slope

asymptotic densities of lozenges near $(t,x)$.

**Proposition** $\Omega(t,x)$ is related to $f(t,x)$ by

$$(***) \qquad f(t,x) = \frac{\Omega - x}{\Omega - x + t} = 1 - \frac{t}{\Omega - x + t}$$

$\Omega \in \mathbb{C}^+$, $f \in \mathbb{C}^+$

[No proof yet] Wait for Lecture 3

Corollary | $\Omega(t,x)$ is a diffeomorphism between liquid region (interior of circle) and $\mathbb{C}^+$ such that

$$f(t,x) = F(\Omega(t,x))$$

complex slope

complex coordinate in $\mathbb{C}^+$

holomorphich and not depending on $(t,x)$

Proof $\Omega$ solves $(2-t)\Omega^2 + (1-2x)\Omega + (t-x) = 0$

Using $(***)$ it is $2f \cdot \Omega - (1-f)\Omega^2 + 1 = 0$

$$f = \frac{\Omega^2 - 1}{\Omega^2 + 2\Omega} \qquad \blacksquare$$

Observation |. Properties of Corollary uniquely define $\Omega$ up to Moebius transformations of the upper-halfplane

$$\Omega \to \frac{a + b\Omega}{c + d\Omega}$$

„Riemann uniformization theorem"

Covariance of GFF $-\frac{1}{2\pi} \ln \left| \frac{z-w}{z-\bar{w}} \right|$ is unchanged under Moebius transformations

Conjecture (Kenyon-Okounkov): For arbitrary simply-

Conjecture (Kenyon-Okounkov): For arbitrary simply-connected large domain, the same global fluctuation theorem holds with $\Omega'$ produced from the Observation.

Limit shape theorem is known for any domains

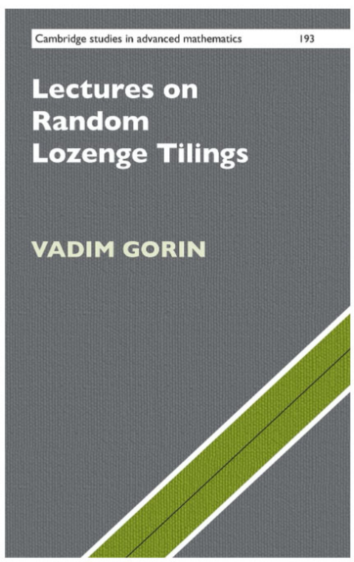
JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 2, Pages 297–346
S 0894-0347(00)00355-6
Article electronically published on November 3, 2000

A VARIATIONAL PRINCIPLE FOR DOMINO TILINGS

HENRY COHN, RICHARD KENYON, AND JAMES PROPP

Wide open for general domains

Heuristics:




Lectures 11/12

Next two lectures: prove GFF for hexagon

The Annals of Probability
2015, Vol. 43, No. 1, 1–43
DOI: 10.1214/12-AOP823
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ASYMPTOTICS OF UNIFORMLY RANDOM LOZENGE TILINGS OF POLYGONS. GAUSSIAN FREE FIELD[1]

BY LEONID PETROV

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 1, January 2017, Pages 27–66
http://dx.doi.org/10.1090/jams/854
Article electronically published on January 28, 2016

CENTRAL LIMIT THEOREMS
FOR BIORTHOGONAL ENSEMBLES AND ASYMPTOTICS
OF RECURRENCE COEFFICIENTS

JONATHAN BREUER AND MAURICE DUITS

The Annals of Probability
2018, Vol. 46, No. 3, 1279–1350
https://doi.org/10.1214/17-AOP1185
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ON GLOBAL FLUCTUATIONS FOR NON-COLLIDING PROCESSES[1]

BY MAURICE DUITS

Advances in Mathematics
Volume 338, 7 November 2018, Pages 702-781

Fluctuations of particle systems determined by Schur generating functions

Alexey Bufetov [a], Vadim Gorin [a b]

arXiv > math > arXiv:2205.15785

**Mathematics > Probability**

*[Submitted on 31 May 2022]*

**Dynamical Loop Equation**

Vadim Gorin, Jiaoyang Huang

← approach for our lectures

stat.berkeley.edu/~vadicgor/teaching.html

**Vadim Gorin** | About me | Research | Integrable FRG | Teaching

Lecture 1, Lecture 2, Lecture 3

Problem set 1, Problem set 2

In Summer 2023 I am giving a mini-course on **Gaussian Free Field in random lozenge tilings** at CUNY Dimers summer school.