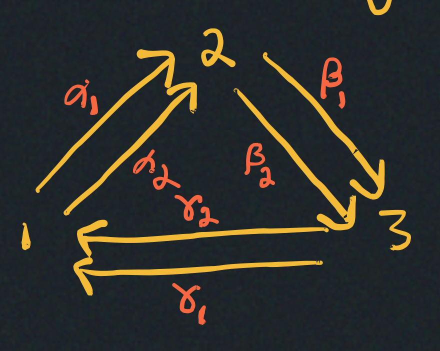
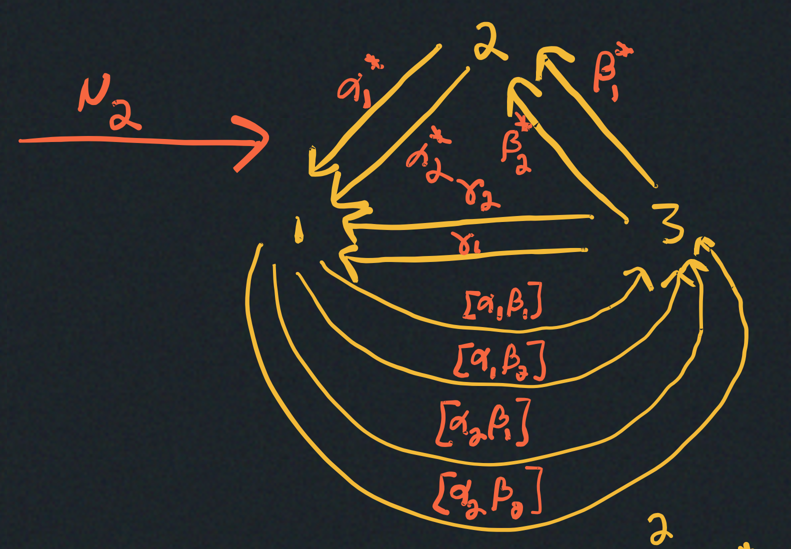
Exercise Solutions

1. Show nonrigid: Consider the cycle $\alpha_1\beta_2\gamma_1\alpha_2\beta_1\gamma_2$

Show Nondegenerate: Try a mutation!





$W = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2$

$$\tilde{W} = \underbrace{[\alpha_1\beta_1]\gamma_1 + [\alpha_2\beta_2]\gamma_2}_{[W]} + \underbrace{\sum_{i,j=1}^{2} \beta_i^*\alpha_j^*[\alpha_j\beta_i]}_{\Delta_2}$$

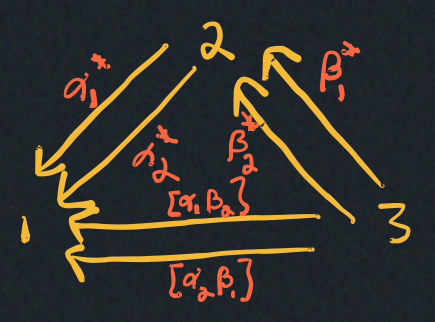
Taking potential at ..... γ_1 gives $[\alpha_1\beta_1]=0$

γ_2 gives $[\alpha_2\beta_2]=0$

$[\alpha_1\beta_1]$ gives $\gamma_2=0$

$[\alpha_2\beta_2]$ gives $\gamma_1=0$

Now we adjust $(\tilde{Q},\tilde{W})$ accordingly to get an equivalent $(\tilde{Q}',\tilde{W}')$



$(\tilde{W})' = \beta_1^*\alpha_2^*[\alpha_2\beta_1] + \beta_2^*\alpha_1^*[\alpha_1\beta_2]$

Isomorphic to (Q,W)! Any further mutation will result in an isomorphic quiver with potential by symmetry, so we can never create digons.

2a) $A_3 = \mathbb{k}(1\to 2\to 3)$

(simple representation at 1)

$$\begin{tikzcd}[column sep=small]
(S(1), P(2)\oplus P(3)) \arrow[drr, dash, "\mu_1"'] & & (S(2), P(1)\oplus P(3)) \arrow[d, dash, "\mu_2"] & & (S(3), P(1)\oplus P(2)) \arrow[dll, dash, "\mu_3"] \\
& & (0, P(1)\oplus P(2)\oplus P(3)) & &
\end{tikzcd}$$

3a) More general:

Mutating $(0, \bigoplus_{i\in[n]} P(i))$ at k gives $(S(k), \bigoplus_{i\neq k} P(i))$ where $S(k)$ is the simple representation at k which has a copy of the field $\mathbb{k}$ at index k and 0's everywhere else. We know this because $S(k)$ is the only indecomposable representation M satisfying $\mathrm{Hom}(\bigoplus_{i\neq k} P(i), M)=0$, which is a condition of τ-rigidity.

3b) $\chi(Gr_{\underline{e}}(S(k))) = \begin{cases} 1 & e=(0,\dots,0) \text{ or } e=(0,\dots,1,\dots,0) \\ 0 & \text{else} \end{cases}$

1 in k^{th} place
0 elsewhere

Injective resolution of $S(k)$:

$$0 \hookrightarrow S(k) \longrightarrow I(k) \hookrightarrow \bigoplus_{\alpha: j\to k} I(j)$$

injective representation at k

$$\Rightarrow \operatorname{ind}(S(k)) = \Big(\sum_{\alpha: j\to k} e_j\Big) - e_k$$

$$\varphi(\mu_k(P(k))) = \varphi(S(k))$$
$$= x^{\operatorname{ind}(S(k))} \cdot \sum_{\underline{e}\in\mathbb{Z}_{\geq 0}^n} \chi(Gr_{\underline{e}}(S(k))) \prod_{i=1}^{n} x_i^{\langle S(i), \underline{e}\rangle}$$

$$= x^{\left(\left(\sum_{\alpha: j\to k} e_j\right) - e_k\right)} \Bigg(\underbrace{1}_{\underline{e}=(0,\dots,0)} + \underbrace{x^{\left(\sum_{k\to i} e_i - \sum_{j\to k} e_j\right)}}_{\underline{e}=(0,\dots,1,\dots,0)} \Bigg)$$

$$= \left(\frac{\prod_{j\to k} x_j}{x_k} \right) \left(1 + \frac{\prod_{k\to i} x_i}{\prod_{j\to k} x_j} \right)$$

$$= \frac{\prod_{j\to k} x_j + \prod_{k\to i} x_i}{x_k} = \mu_k(x_1,\dots,x_n) = \mu_k\Big(\varphi\big(0, \bigoplus_{i\in[n]} P_i\big)\Big)$$