Dimers
CUNY

Statistical physics and probability

Vadim Gorin, TA: Matthew Nicoletti **Discussion 1**

Outline

Today we will review some questions about the Gaussian Free Field (GFF).

First, we briefly recall the definition. Fix a domain $D \subset \mathbb{R}^2 = \mathbb{C}$. Using complex variables $z = x + \mathrm{i}y$ and $z' = x' + \mathrm{i}y'$, let $G_D(z, z')$ denote the Dirichlet greens function on D. This means that

$$-\Delta_z G_D(z, z') = \delta(z - z')$$

subject to the boundary condition

$$G(z, z') \to 0 \qquad \text{as } z \to \partial D.$$

Above, $\Delta_z = \partial_x^2 + \partial_y^2$ is the Laplacian on D with respect to z.

The Dirichlet GFF on D is a Gaussian process $\mathscr{G}_D$ indexed by measures μ on D with the property that

$$\int_D \int_D G_D(z, z') \mu(dz) \mu(dz') < \infty.$$

The random distribution $\mathscr{G}_D$ is defined by the property that for any such measures $\mu_1, \dots, \mu_k$, the pairings $\langle \mathscr{G}_D, \mu_i \rangle$ are jointly Gaussian with mean 0 and covariance

$$\mathbb{E}\left[\langle \mathscr{G}_D, \mu_i \rangle \langle \mathscr{G}_D, \mu_j \rangle\right] = \int_D \int_D G_D(z, z') \mu_i(dz) \mu_j(dz'). \tag{1}$$

In two dimensions, the Gaussian free field is not a function, but rather it is a distribution. In particular, its value at a point is not well defined. Still, however, it is useful to think that heuristically

$$\langle \mathscr{G}_D, \mu \rangle = \int_D \mathscr{G}_D(z) \mu(dz)$$

and that the defining property (1) of the GFF is equivalent to

$$\mathbb{E}\left[\mathscr{G}_D(z) \mathscr{G}_D(z')\right] = G_D(z, z').$$

1. **1d GFF warmup.** A *Brownian bridge* $B(t)$ can be defined as

- a Brownian motion conditioned to equal 0 at time 1
- the Gaussian process on $[0, 1]$ with covariance

$$\mathscr{C}(s, t) = \min(s, t)(1 - \max(s, t)).$$

Show that $B(t)$ defines a 1d GFF. I.e. show that

$$\partial_t^2 \mathscr{C}(s, t) = -\delta(t - s)$$

and that $\mathscr{C}(s, t)$ satisfies Dirichlet boundary conditions.

2. **GFF in $\mathbb{H}$.** Show that in the upper half plane, $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) \geq 0\}$, we have

$$G_{\mathbb{H}}(z, z') = -\frac{1}{2\pi} \log \left| \frac{z - z'}{z - \overline{z}'} \right|.$$

3. **Pullback of GFF (as in random tilings).** Consider lozenge tilings of the $N \times N \times N$ regular hexagon. Then, as discussed in the lecture, the height function fluctuations,

$$H_N(t, x) = h_N(\lfloor Nt \rfloor, \lfloor Nx \rfloor) - \mathbb{E}[h_N(\lfloor Nt \rfloor, \lfloor Nx \rfloor)$$

converge to a certain Gaussian field $\mathscr{G}$ in a certain domain $\mathscr{L}$, known as the *liquid region* or *rough region*.

However this Gaussian field $\mathscr{G}$ is not a *free field*; it is the *pullback* of a Gaussian free field in the upper half plane under a certain diffeomorphism $\Omega : \mathscr{L} \to \mathbb{H}$. We have

$$H_N(t, x) \to \mathscr{G}(t, x) = \mathscr{G}_{\mathbb{H}}(\Omega(t, x))$$

where

$$\Omega(t, x) = \frac{1 - 2x - \sqrt{1 - 8t + 4t^2 + 4x - 4tx + 4x^2}}{2(-2 + t)}$$

and $\mathscr{L} = \{1 - 8t + 4t^2 + 4x - 4tx + 4x^2 < 0\}$. (Note that in (t, x) coordinates, this region is bounded by an ellipse, rather than a circle.)

Compute the distribution of the pairing $\langle \mathscr{G}, x\delta_{\mathrm{L}} \rangle$ where $\delta_{\mathrm{L}}(dt, dx) = \delta(t - 1)dtdx$ is the delta function on the vertical line $\mathrm{L} = \{t = 1\}$ shown in the Figure below, and x is the x coordinate. More precisely, we define $x\delta_{\mathrm{L}}(dt, dx)$ by the property that, for any smooth function $u(t, x)$,

$$\langle u, x\delta_{\mathrm{L}} \rangle = \int_{\mathscr{L}} u(t, x) x \delta_{\mathrm{L}}(dt, dx) = \int_{\mathrm{L} \cap \mathscr{L}} u(1, x) x \, dx.$$