Field/String duality in Turbulence

Alexander Migdal
Department of Physics, New York University
726 Broadway, New York, NY 10003

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Slide Title

Abstract

We summarize and rethink as string/field duality the recent progress we made in describing the 3D turbulence as the low-temperature limit of the Gibbs statistics of vortex surfaces. The talk is addressed to the String Field community and the Fluid Dynamics community in the hope that they join forces in solving the ancient problem of turbulence.

The Enigma of Turbulence

For the last 200 years, people are trying to solve a simple set of equations

$$\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} = \nu \nabla^2 \vec{v} + \vec{\nabla} p;$$
$$\vec{\nabla} \cdot \vec{v} = 0;$$

The only parameter is the viscosity ν which should tend to zero while keeping finite dissipation of energy

$$\mathcal{E} = \nu \int d^3 r \vec{\omega}^2;$$
$$\vec{\omega} = \vec{\nabla} \times \vec{v};$$

We shall call this limit a turbulent limit. It makes the problem universal and tantalizingly simple.

The Enigma of Turbulence

It is known what happens in the turbulent limit, but we do not know how to explain this and how to describe it quantitatively.

Instead of a unique solution depending smoothly on the initial data, or some unique fixed point, we have a fixed manifold– statistical distribution of vorticity structures with some universal properties.

We do not even know the physical origin of this distribution, not to mention its complexity and its multi-fractal properties, far more complex than CFT.

The CFT is inapplicable here because of nonlocal effects. Conservation law $\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \vec{\omega} = 0$ would require conformal dimension $d - 1 = 2$ for both velocity and vorticity which is impossible mathematically and wrong experimentally.

The Enigma of Turbulence

If you take a fresh look at the expression for energy dissipation, it becomes clear that the vorticity $\vec{\omega}$ should be singular in some regions of space to compensate for the infinitesimal viscosity in front.

Such singularities are known to exist, in particular in liquid helium, where the viscosity is zero. These are vortex lines and vortex surfaces.

Vortex lines have infinite velocity, but for the vortex surface the velocity is finite. One can imagine the smeared velocity discontinuity, creating a large vorticity, such that its square would compensate the viscosity in front.

As we shall shortly see, this is exactly what happens, but there are some interesting details to work out ([1], [2]).

The Enigma of Turbulence

To give an idea how the the vortex surface explains the dissipation, let us consider a local vicinity $z \to 0$ of some point $(x, y, 0)$ on vortex surface and look at the balance of the singular terms, involving the normal derivatives of tangent components of velocity $\vec{v}_t$

$$v_z \partial_z \vec{v}_t = \nu \partial_z^2 \vec{v}_t$$

For the solution to be finite in both directions, we need $v_z(0) = 0$, then [3], [4]

$$\vec{\omega} \propto \partial_z \vec{v}_t \propto N(z, h);$$
$$\vec{v}_t \propto \frac{1}{2} \Delta \vec{v}_t \operatorname{erf}\left(\frac{z}{h\sqrt{2}}\right);$$
$$h = \sqrt{\frac{\nu}{-\partial_z v_z}}$$

where $N(z, h)$ is the normal distribution with variance h^2.

The Enigma of Turbulence

Now the enstrophy will involve the square of this Gaussian function

$$\nu\vec{\omega}^2 \propto \nu(\Delta\vec{v}_t)^2 N(z,h)^2 = \frac{\sqrt{\nu}}{2\sqrt{\pi}}\sqrt{-\partial_z v_z}(\Delta\vec{v}_t)^2 N\left(z, \frac{h}{\sqrt{2}}\right);$$

In the limit $\nu \to 0, h \to 0$, we get a surface integral

$$\mathcal{E} = \frac{\sqrt{\nu}}{2\sqrt{\pi}}\int_S (\Delta\vec{v}_t)^2\sqrt{-\partial_z v_z}$$

This is the viscosity anomaly([1]), with the new factor $\sqrt{-\partial_z v_z}$ which was assumed constant in [1] .

The Enigma of Turbulence

The field/string duality we are advocating here starts from the following idea. The dynamics and statistics of vortex surfaces in the turbulent limit are defined by the Euler equation, if we only take into account the anomalous dissipation.

The strongly fluctuating velocity and singular vorticity can be described by a dual dynamics and statistics of vortex surfaces in the same way as the fluctuating gauge field is described by a fluctuating geometry in ADS-CFT correspondence.

The Duality in QFT has the strong coupling phase of the fluctuating gauge field corresponding to the weak coupling phase of the dual string theory and vice versa.

With some significant distinctions, the same kind of field-string duality exists, as we believe, in the turbulence.

Dynamics of Vortex Surfaces

Let us now describe the dynamics of vortex surfaces ([5], [6]).

The following ansatz describes the vortex surface vorticity:

$$\vec{\omega}(\vec{r}) = \int_{\Sigma} d\vec{\Omega}\delta^3\left(\vec{X} - \vec{r}\right)$$

where the 2-form

$$d\vec{\Omega} \equiv d\Gamma \wedge d\vec{X} = d\xi_1 d\xi_2 e_{ab} \frac{\partial \Gamma}{\partial \xi_a} \frac{\partial \vec{X}}{\partial \xi_b}$$

The conservation of vorticity

$$\vec{\nabla} \cdot \vec{\omega} = 0;$$

is built into this ansatz for arbitrary $\Gamma(\xi)$.

Dynamics of Vortex Surfaces

The function $\Gamma(\xi)$ is defined modulo diffeomorphisms $\xi \Rightarrow \eta(\xi)$ and is conserved in Lagrange dynamics with velocity expressed in terms of the vorticity surface by Biot-Savart law:

$$\partial_t \Gamma = 0;$$

$$\partial_t \vec{X} = \vec{v}(\vec{X});$$

$$\vec{v}(\vec{r}) = \vec{\nabla} \times \vec{\Psi}(\vec{r});$$

$$\vec{\Psi}(\vec{r}) = -\int \frac{d\vec{\Omega}(\vec{X})}{4\pi|\vec{r} - \vec{X}|}$$

Dynamics of Vortex Surfaces

This function is related to the velocity discontinuity

$$\Gamma = \int \Delta\vec{v} \cdot d\vec{r};$$

$$\Delta\vec{v} = \vec{v}\left(\vec{X}^{+}\right) - \vec{v}\left(\vec{X}^{-}\right)$$

The line integral does not depend on the path but only on its end point, due to the lack of the vorticity flux through the surface.

Dynamics of Vortex Surfaces

The Lagrange equations of motion for the surface were shown to follow from the action

$$S = \int \Gamma dV - \int H dt;$$

$$dV = d\xi_1 d\xi_2 dt \frac{\partial \vec{X}}{\partial \xi_1} \times \frac{\partial \vec{X}}{\partial \xi_2} \cdot \partial_t \vec{X};$$

$$\mathcal{H} = \frac{1}{2} \int d^3 r \vec{v}^2 = \frac{1}{2} \int_S \int_S \frac{d\vec{\Omega} \cdot d\vec{\Omega}'}{4\pi |\vec{X} - \vec{X}'|};$$

dV is the 3-volume swept by the surface area element in its movement for the time dt.

Dynamics of Vortex Surfaces

In addition to the conserved energy (which conservation is broken by the viscosity anomaly), we also have conserved momentum

$$\vec{P} = \int d^3\vec{r}\vec{v} = \frac{1}{3}\int \Gamma d\vec{\sigma};$$
$$d\vec{\sigma} = d^2\xi \partial_1 \vec{X} \times \partial_2 \vec{X};$$

One can observe that the particular $\Gamma = \Gamma^*[\vec{X}]$ minimizing this Hamiltonian at fixed $\vec{X}$ provides a stationary solution of the Lagrange-Euler equations, with the normal velocity vanishing on the surface.

Dynamics of Vortex Surfaces

In the case of the handle H on a surface, Γ acquires extra term $\Delta\Gamma = \oint_\gamma \Delta\vec{v} \cdot d\vec{r}$ when the point goes around one of the cycles $\gamma = \{\alpha, \beta\}$ of the handle.

This $\Delta\Gamma$ does not depend on the path shape because there is no normal vorticity at the surface, and thus there is no flux through the surface. This topologically invariant $\Delta\Gamma$ represents the flux through the handle cross-section.

Dynamics of Vortex Surfaces

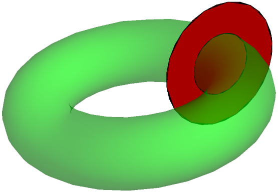


Figure: The flux through the red disk reduces to $\Delta\Gamma = \oint_\gamma \Delta\vec{v} \cdot d\vec{r}$

Dynamics of Vortex Surfaces

As a consequence, the velocity circulation around an arbitrary contour C in 3D space, avoiding these surfaces, reduces to the algebraic sum of such fluxes for all the handles linked to C.

This ambiguity in Γ makes our action multivalued as well.

Stable Vortex Surfaces

The turbulence phenomenon starts with the instability of the steady solutions of the Euler equation.

We expect it to eventually cover a certain stable manifold, like the energy surface in an ergodic dynamical system.

In the turbulence case, this manifold should involve the constant energy dissipation.

Let us consider a steady closed vortex surface $\mathcal{S}$, to check whether it can belong to this stable manifold.

In the outside space $\mathcal{S}^+ : \partial\mathcal{S}^+ = \mathcal{S}$ there is no vorticity, so the flow can be described by a potential

$$v_\alpha(\vec{r}) = \partial_\alpha \Phi^+(\vec{r});\ \forall \vec{r} \in S^+;$$

$$\partial_\alpha v_\alpha(\vec{r}) = \partial_\alpha^2 \Phi^+(\vec{r}) = 0;\,;\ \forall \vec{r} \in \mathcal{S}^+$$

$$v_n(\vec{r}) = \partial_n \Phi^+(\vec{r}) = 0;\ \forall \vec{r} \in \mathcal{S}$$

Stable Vortex Surfaces

The steady Euler equation

$$v_\beta \partial_\beta v_\alpha + \partial_\alpha p = 0;$$
$$\partial_\alpha v_\alpha = 0;$$

is satisfied with this potential flow, given the pressure p satisfies the Bernoulli equation

$$p^+(\vec{r}) = -\frac{1}{2}\left(\partial_\alpha \Phi^+(\vec{r})\right)^2; \; \forall \vec{r} \in S^+;$$

The flow inside the volume $\mathcal{S}^-$ bounded by the surface is stagnant: nothing comes through the surface, so the velocity must be zero, and so is the potential and pressure inside.

$$p^-(\vec{r}) = \Phi^-(\vec{r}) = 0; \; \forall \vec{r} \in S^-;$$

Stable Vortex Surfaces

Here is an example of such a potential flow corresponding to the spherical vortex surface

$$\Phi^{+}(\vec{r}) = \left(\frac{1}{2} + \frac{1}{3|\vec{r}|^{5}}\right)\left(a(x^2 - z^2) + b(y^2 - z^2)\right);$$

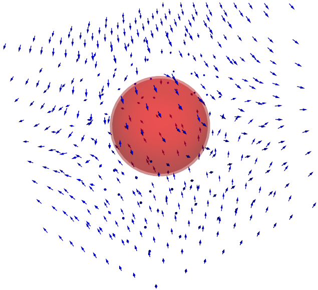


Figure: Spherical Vortex surface with vanishing normal velocity

Stable Vortex Surfaces

The normal velocity vanishes on the surface, but the tangent velocity is finite – the fluid slides along the surface. As there is no velocity inside, this creates the tangent discontinuity $\Delta v_i, i = x, y$, and the corresponding delta function for vorticity

$$\Delta \vec{v}_i(\vec{r} \in \mathcal{S}) = \vec{\nabla}_i \Gamma(\vec{r});$$
$$\Gamma(\vec{r}) = \Phi^+(\vec{r} \in \mathcal{S});$$
$$\omega_i = -e_{ij}\partial_j \Gamma(\vec{r})\delta(z),\ \omega_z = 0;$$

From here, using incompressibility, we arrive at an important relation

$$-\partial_z v_z^+ = \partial_i v_i^+ = \vec{\nabla}_i^2 \Gamma(\vec{r} \in \mathcal{S})$$

For the above example

$$\Gamma(\vec{r}) = \frac{5}{6}\left(a(x^2 - z^2) + b(y^2 - z^2)\right);\ \forall x^2 + y^2 + z^2 = 1;$$

Stable Vortex Surfaces

The Lagrange dynamics of an ideal fluid tells us that every point moves with local velocity. This also applies to the vortex surfaces. In the local tangent plane, x, y

$$\partial_t z(x,y,t) = v_z\left(x, y, z(x,y,t)\right)$$

The small deviation δz from the steady shape

$$\mathcal{S} : z = z_S(x,y); v_z\left(x, y, z_S(x,y)\right) = 0$$

will exponentially grow (Kelvin-Helmholtz instability) or decay with time

$$\delta z(x,y,t) \propto \exp\left(t \partial_z v_z\left(x, y, z_S(x,y)\right)\right)$$

The instability is absent if $\partial_z v_z\left(x, y, z_S(x,y)\right) < 0$.

Stable Vortex Surfaces

This stability requirement is NOT satisfied for the above example. The tangent Laplacian is not positive definite

$$\vec{\nabla}_t^2 \Gamma = -\Gamma = -\frac{5}{6}\left(a(x^2 - z^2) + b(y^2 - z^2)\right)$$

Therefore, this solution is steady but not stable.

Our next goal is to compare the Euler and Navier-Stokes equations and find the stable subset of the phase space $\Gamma, \vec{X}$ of the vortex surfaces.

For this purpose, we need to match the vortex surface ansatz with the exact solution of the Navier-Stokes equation in a thin boundary layer.

Stable Vortex Surfaces

It was recently observed [1] that in addition to the Burgers-Townsend sheet with the symmetric Gaussian profile of vorticity, there is an asymmetric solution, expressed in the Hermite function with the negative fractional index.

This asymmetric solution decays as a Gaussian on one side but only as a power on the other side of the sheet.

In other words, vorticity leaks from that sheet, unlike the Burgers-Townsend sheet where it was confined to the thin layer.

Later, another important observation was made [7]. The asymmetric sheet turned out to be the general solution of the Navier-Stokes equation for the constant strain

$$S_{\alpha\beta} = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha)$$

Stable Vortex Surfaces

The eigenvalues of the strain add up to zero in virtue of incompressibility, so there are two independent parameters here.

$$\hat{S} = \text{diag}\,(\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$$

We can always assume that the eigenvalues are sorted in decreasing order. The Kelvin-Helmholtz stability demands that $\lambda_1 + \lambda_2 > 0$.

The asymmetric solution exists when

$$\lambda_1 \geq \lambda_2 > 0;$$

The special case considered in [1] corresponds to $\lambda_1 = \lambda_2 > 0$.

Stable Vortex Surfaces

The vorticity of the generic solution is proportional to Hermite function

$$\omega \propto \exp\left(-\frac{z^2}{2h^2}\right) H_\mu\left(\frac{z}{h\sqrt{2}}\right);$$

$$\mu = -\frac{\lambda_2}{\lambda_1 + \lambda_2};$$

$$\omega(z \to +\infty) \propto (z)^\mu \exp\left(-\frac{z^2}{2h^2}\right);$$

$$\omega(z \to -\infty) \propto (-z)^\mu$$

There is also a mirror solution with $z \Rightarrow -z$.

Stable Vortex Surfaces

For every finite λ_1, λ_2 vorticity decays at least on one side as a negative fractional power $|z|^\mu; \mu < 0$, which makes it unacceptable for the vortex surface statistics.

The Burgers-Townsend solution corresponds to the exceptional case $\lambda_1 > 0, \lambda_2 = 0$. The solution reads

$$\vec{v} = \{ax, bS_h(z), -az\};$$

$$S^0_{\alpha\beta} = \text{diag}\,(a, 0, -a)\,;$$

$$\vec{\omega} = \{-bS'_h(z), 0, 0\};$$

$$S'_h(z) = \frac{1}{h\sqrt{2\pi}} \exp\left(-\frac{z^2}{2h^2}\right);$$

$$S_h(z) = \operatorname{erf}\left(\frac{z}{h\sqrt{2}}\right);$$

$$a = \frac{\nu}{h^2};$$

Stable Vortex Surfaces

In this case, the vorticity becomes Gaussian, and the velocity gap becomes an error function. In the limit of $h \to 0$ the vorticity reduces to $\delta(z)$, and velocity gap reduces to $\mathrm{sign}(z)$.

These solutions for various ratios μ of eigenvalues were investigated in [7]. An interesting case is negative λ_2 (super-Townsend in [7]). In that case $\mu < -1$ so that power decay is even stronger that in case of positive λ_2.

The study of time evolution in each of these solutions showed a decay to zero for the asymmetric case $\lambda_2 > 0$, and instability for the super-Townsend case $\lambda_2 < 0$. Only in the Burgers-Townsend case $\lambda_2 = 0$ there was a stable solution which did not need any supply of vorticity from $\pm\infty$.

Stable Vortex Surfaces

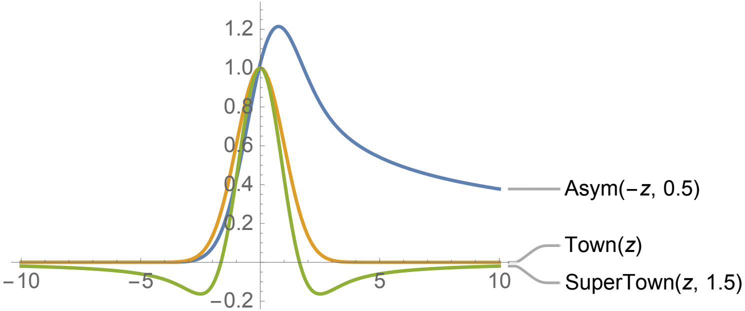
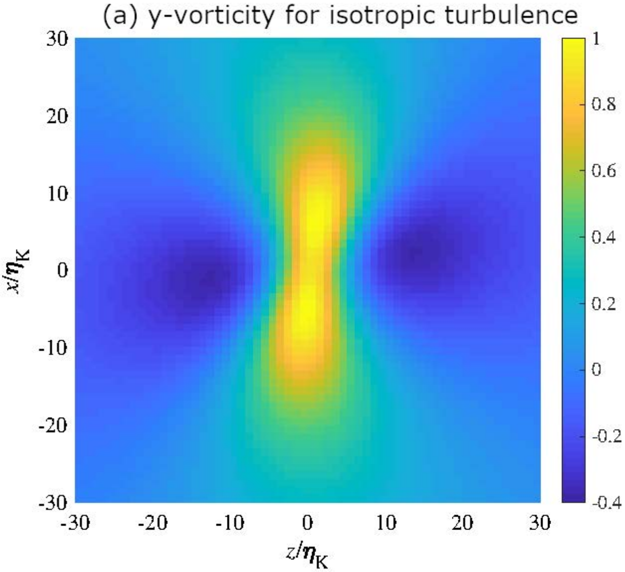


Figure: The vorticity profiles for asymmetric, Townsend and super-Townsend strains

Stable Vortex Surfaces



Stable Vortex Surfaces

We reach the following radical conclusion: the stable vortex surfaces only exist when the matrix of second derivatives of Γ satisfies two conditions:

$$\det ||\partial_i \partial_j \Gamma|| = 0;$$
$$\partial_i^2 \Gamma > 0;$$

The meaning of the first condition is that Γ depends on only one of the two coordinates in the tangent plane, up to a global linear transformation of these coordinates.

The second condition means that the planar Laplacian of Γ must be positive.

Stable Vortex Surfaces

In the simplest spherical topology, one can parametrize the surface by two Euler angles on a sphere S_2.
The function of the azimuth φ is excluded by the requirement of the Kelvin-Helmholtz stability (there are no periodic functions with positive second derivative).
The dependence of the polar angle $\Gamma(\theta, \varphi) = \gamma(\theta)$ is allowed, as there is no periodicity in θ.
To provide the vanishing normal velocity for an arbitrary azimuth φ tin general case of asymmetric tensor $W_{\alpha\beta}$ with $\lambda_1 \neq \lambda_2$ there should be some nontrivial asymmetry of its shape :

$$\xi = (\theta, \varphi);$$
$$\vec{n}(\xi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta);$$
$$\vec{X}(\xi) = \rho(\cos\theta, \varphi)\vec{n}(\xi);$$
$$\Gamma(\xi) = \gamma(\cos\theta);$$

Stable Vortex Surfaces

The condition of positive Laplacian in these variables can be exactly resolved

$$\vec{\nabla}_i^2\Gamma = (1-z^2)\gamma''(z) - 2z\gamma'(z) = 2e^{p(z)};$$
$$\gamma(z) = 2\,\mathrm{arctanh}(z)\int_1^z e^{p(t)}\,dt -$$
$$e^{p(z)}\left(\log(1-z^2) + 2z\,\mathrm{arctanh}(z)\right)$$

This defines a stable subset of the phase space.

$$\Gamma(\theta,\varphi), \vec{X}(\theta,\varphi) \Rightarrow p(\cos\theta), \rho(\cos\theta,\varphi)$$

Stable Vortex Surfaces

We have solved numerically the equation $v_n = 0$ together with the determinant equation $\det S = 0$ on a sphere.
The plot of the corresponding functions $\gamma(\cos(\theta)), \lambda = \vec{\nabla}^2\gamma$ is presented here

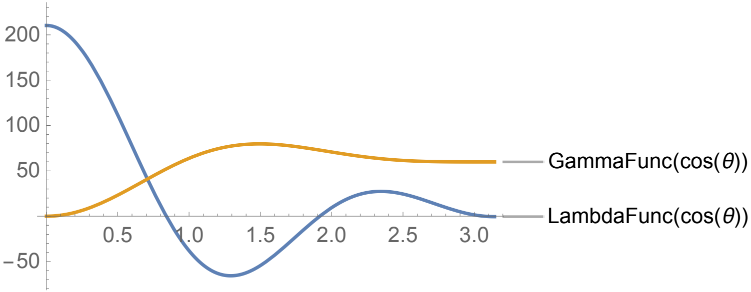


Figure: $\gamma(cos\theta), \lambda(\cos\theta)$, with first 20 harmonics minimizing the Hamiltonian

Conservation of Enstrophy

For a consistency of this model of turbulence, the energy dissipation itself must be conserved – only then we would have the stationary process.

We also have to provide some source of energy coming through the boundary conditions (large scale motions).

The energy pumping rate $\mathcal{W}$ must match the energy dissipation rate $\mathcal{E}$ and therefore it also must be conserved.

Conservation of Enstrophy

Let us study the general formula for the time derivative of the energy dissipation in the Navier-Stokes equation

$$\partial_t \mathcal{E} = 2\nu \int d^3 r \, \cancel{v_\beta \partial_\beta \left(\frac{1}{2}\omega_\alpha^2\right)} - \omega_\alpha \omega_\beta \partial_\beta v_\alpha;$$

$$\mathcal{E} = \frac{\sqrt{\nu}}{2\sqrt{\pi}} \int_{\mathcal{S}} dS \left(\vec{\nabla}\Gamma\right)^2 \sqrt{\vec{\nabla}^2 \Gamma};$$

$$\partial_t \mathcal{E} = 0;$$

The conservation of dissipation in our reduced vortex sheet dynamics is essential for the steady energy flow.

Energy Pumping

We view our closed vortex surface as part of the "ideal gas" of vortex bubbles with low density such that we can neglect their collision. This idealization can later be removed, but for now we study such an ideal gas of vortex bubbles of various shapes and strengths.

We assume that the thermostat of remaining (large) vortex structures does not create mean velocity (or we are studying our vortex shell in a Galilean frame where this velocity vanishes).

The next invariant we can fix as a boundary condition for our velocity is a constant strain $W_{\alpha\beta}$, by adding a linear term to the Biot-Savart integral

$$v_\alpha = W_{\alpha\beta} r_\beta - \vec{\nabla} \times \int \frac{d\vec{\Omega}(\vec{X})}{4\pi|\vec{r} - \vec{X}|};$$

$$W_{\alpha\alpha} = 0;$$

Energy Pumping

This tensor is spatially uniform, but could be a particular realization of a Gaussian random symmetric traceless tensor. Remarkably, the presence of a uniform constant strain breaks the time-reversal symmetry, but not the parity, as the strain is parity even.

The eigenvalues $a, b, -a-b$ of the random Gaussian symmetric traceless matrix are distributed as

$$dP(a,b) = \\ |a-b||2a+b||2b+a| da db \exp\left(-\frac{a^2+b^2+ab}{\sigma^2}\right);$$

We assume that with the positive $a+b$ the normal strain $\pm\lambda$ will be negative; otherwise, we redefine the sign of potential.

Energy Pumping

The master equation $v_n = 0$ becomes linear integral equation for Γ (with $\vec{N}(\vec{r})$ being the local normal vector)

$$\vec{N}(\vec{r}) \cdot \vec{\nabla} \times \int_{\mathcal{S}} \frac{d\Gamma \wedge d\vec{r}'}{4\pi|\vec{r} - \vec{r}'|} = \vec{N}(\vec{r}) \cdot W \cdot \vec{r};\ \forall \vec{r} \in \mathcal{S}$$

The variance σ of the boundary strain W enters linearly here, so one can rescale $\Gamma = \gamma\sigma$ after which γ will depend only of the ratio b/a.

Comparing that with the viscosity anomaly we find the scaling law

$$\Gamma \sim \sigma \sim \left(\frac{\mathcal{E}^4}{\nu^2}\right)^{\frac{1}{5}}$$

Energy Pumping

The Wilson loop factor for this field is calculable [8] as a multiple integral over the zero modes of this solution: center of the surface, $O(3)$ rotation matrix needed to diagonalize the strain W and some other auxiliary parameters related to intersection of the loop with the surface.

This solution is similar to the instanton in gauge field theory. At the same time it involves a surface of singular vorticity, and in this aspect it is similar to the string picture of QCD, where the gauge field strength is also confined to the surface.

This analogy does not go further, as the stability condition of the vortex surface significantly reduces the degrees of freedom, effectively making this a one-dimensional theory rather than a full string theory.

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