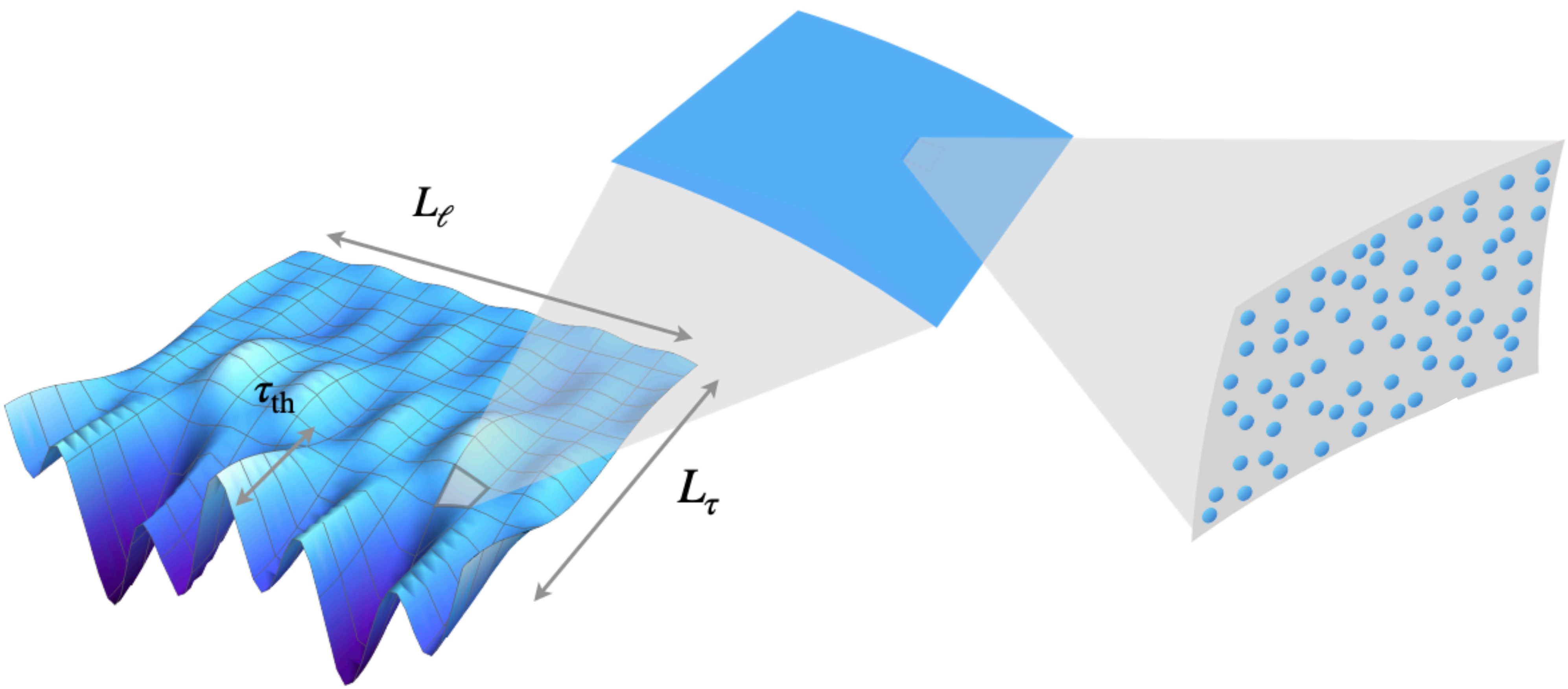
GHD and BBGKY hierarchy



Bruno Bertini

THE ROYAL SOCIETY

University of Nottingham
UK | CHINA | MALAYSIA

Integrability and Integrability Breaking
CUNY, 29 April 2022

Outline:

(I) Quantum Matter Out of Equilibrium and GHD

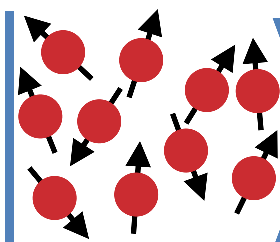
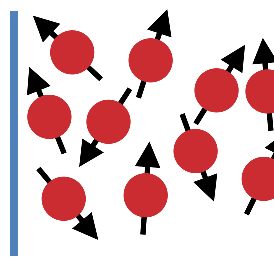
(II) Weakly interacting fermionic quantum gases in 1d and BBGKY hierarchy

(III) Matching the two descriptions

(I) Quantum Matter Out of Equilibrium and GHD

Can we describe interacting many-particle quantum systems out of equilibrium?

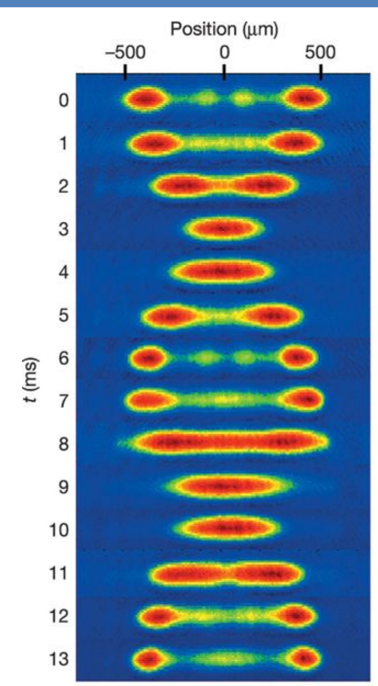
Quantum Quench

$\left|\Psi_0\right\rangle = \Big|$  $\Big\rangle \quad \mapsto \quad \left|\Psi_t\right\rangle = e^{-iHt} \Big|$  $\Big\rangle$

GS of some local H_0

different local Hamiltonian

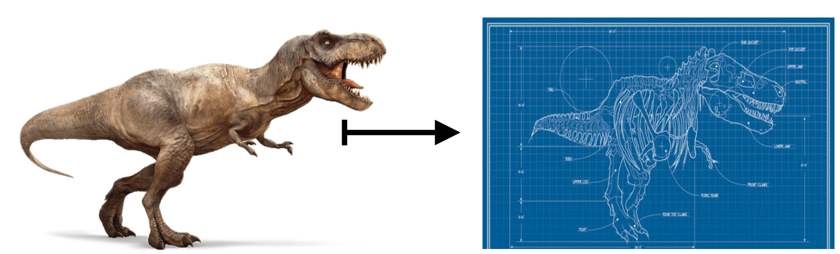
Experimentally Relevant



Kinoshita, Wenger, and Weiss, Nature **400**, 900 (2006).

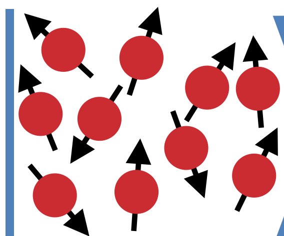
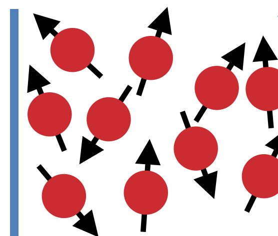
Simple and well defined

Displays the key physics



Can we describe interacting many-particle quantum systems out of equilibrium?

Quantum Quench

$|\Psi_0\rangle = |$  $\rangle \quad \mapsto \quad |\Psi_t\rangle = e^{-iHt}\,|$  $\rangle$

Questions:

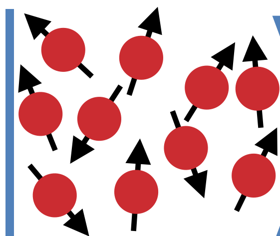
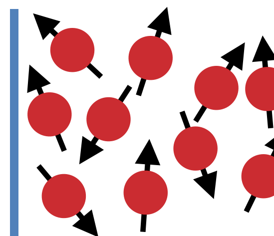
1. How can the system relax despite purely unitary evolution?

2. Thermodynamic/hydrodynamic description of eventual equilibrium?

3. What about the finite time dynamics?

Can we describe interacting many-particle quantum systems out of equilibrium?

Quantum Quench

$$\left| \Psi_0 \right\rangle = \left| \right.$$  $$\left. \right\rangle \mapsto \left| \Psi_t \right\rangle = e^{-iHt} \left| \right.$$  $$\left. \right\rangle$$

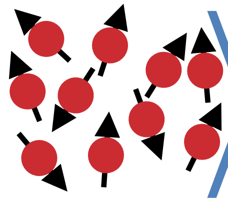
Questions:

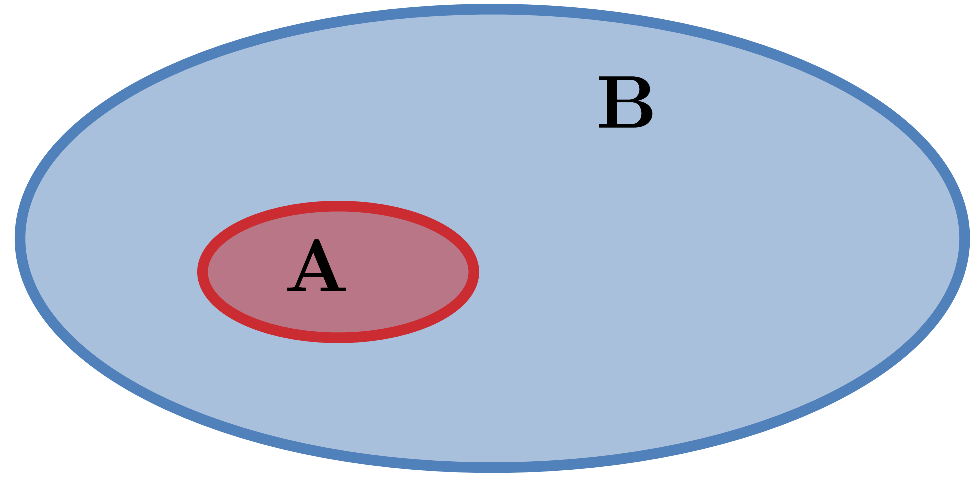
Good understanding

1. How can the system relax despite purely unitary evolution?
2. Thermodynamic/hydrodynamic description of eventual equilibrium?

Open

3. What about the finite time dynamics?

$\left|\Psi_t\right\rangle = e^{-iHt}\Big|$  $\Big\rangle$: large times

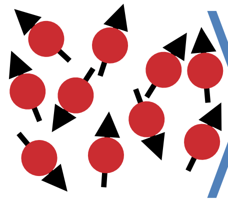


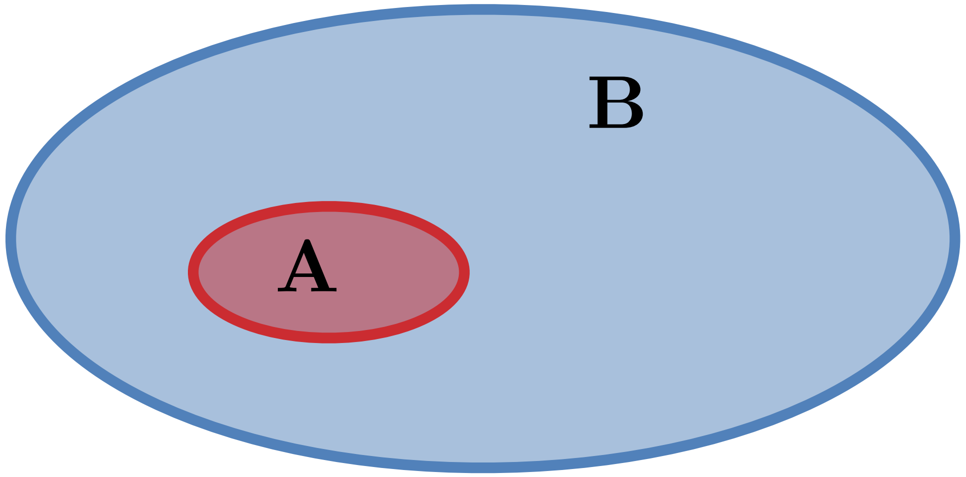
Homogeneous setting

- Closed systems can relax locally

$$\lim_{t\to\infty}\lim_{L\to\infty} \mathrm{tr}_B[\hat{\rho}(t)] = \mathrm{tr}_B[\hat{\rho}_s]$$

Reviews: Polkovnikov, Sengupta, Silva, and Vengalattore, RMP **83**, 863 (2011);
Eisert, Friesdorf and Gogolin, Nat. Phys. **11**, 124 (2015);
Essler and Fagotti, JSTAT 2016 (06), 064002.

$\left| \Psi_t \right\rangle = e^{-iHt} \Big|$  $\Big\rangle$ **: large times**



Homogeneous setting

- Closed systems can relax locally

$$\lim_{t\to\infty} \lim_{L\to\infty} \mathrm{tr}_B[\hat{\rho}(t)] = \mathrm{tr}_B[\hat{\rho}_s]$$

- Stationary state determined by the conservation laws with local density

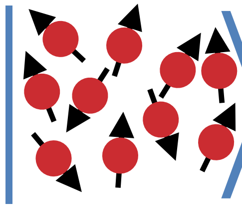
$$\hat{\rho}_s = \frac{1}{Z} e^{-\sum_m \beta_m \hat{Q}^{(m)}} \qquad [\hat{H}, \hat{Q}^{(m)}] = 0 \quad \wedge \quad \hat{Q}^{(m)} = \sum_x \hat{q}_x^{(m)}$$

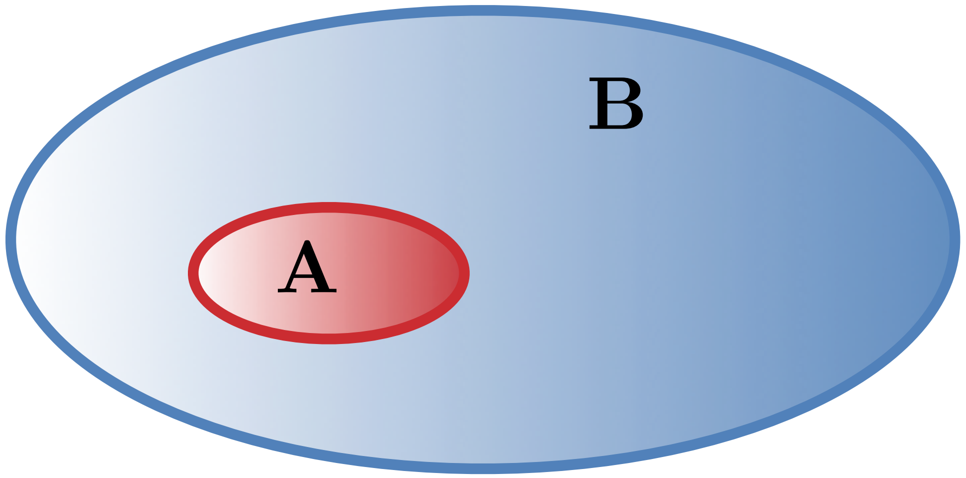
local in space

- Temperatures fixed by

$$\partial_t \hat{q}_x^{(m)}(t) + \partial_x \hat{J}_x^{(m)}(t) = 0 \qquad \longmapsto \qquad \langle \Psi_0 | \hat{q}_x^{(m)} | \Psi_0 \rangle = \mathrm{tr}[\hat{\rho}_s \cdot \hat{q}_x^{(m)}]$$

Reviews: Polkovnikov, Sengupta, Silva, and Vengalattore, RMP **83**, 863 (2011);
Eisert, Friesdorf and Gogolin, Nat. Phys. **11**, 124 (2015);
Essler and Fagotti, JSTAT 2016 (06), 064002.

$\left| \Psi_t \right\rangle = e^{-iHt} \left| \right.$  $\left. \right\rangle$ **: large times**



General setting

- For slow enough variations everything still goes through

$$\hat{\rho}_s = \frac{1}{Z} e^{-\sum_m \beta_m \hat{Q}^{(m)}} \mapsto \hat{\rho}_s(x,t) = \frac{1}{Z} e^{-\sum_m \beta_m(x,t) \hat{Q}^{(m)}}$$

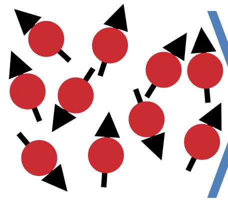
- Temperatures fixed by

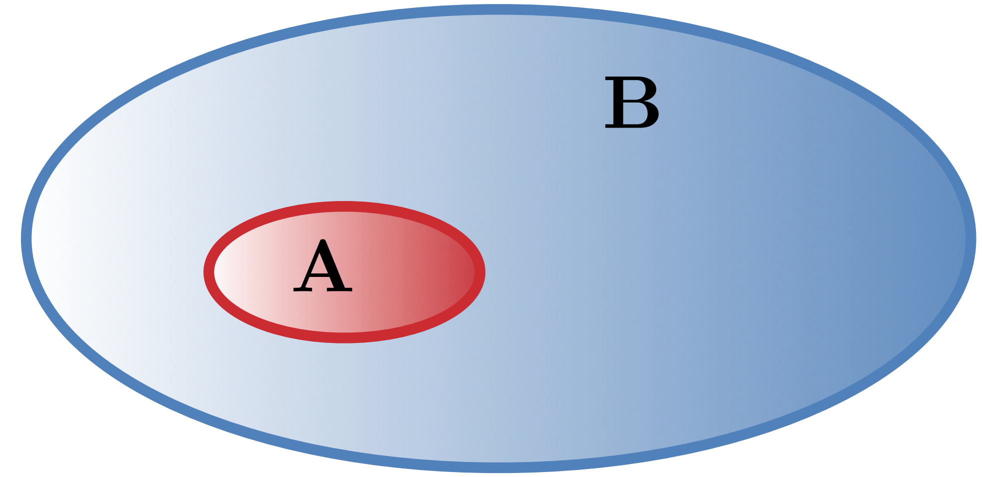
$$O(x,t) := \mathrm{tr}[\hat{O}\,\hat{\rho}_s(x,t)]$$

$$\partial_t \hat{q}_x^{(m)}(t) + \partial_x \hat{J}_x^{(m)}(t) = 0 \quad \longmapsto \quad \partial_t q^{(m)}(x,t) + \partial_x J^{(m)}(x,t) = 0$$

Functionals of $\{\beta_n(x,t)\}$

Reviews: BB, Heidrich-Meisner, Karrasch, Steinigeweg, Prosen, and Žnidarič, RMP **93**, 025003 (2021);
Doyon, SciPost Phys. Lect. Notes **18** (2020);
Alba, BB, Fagotti, Piroli, and Ruggiero, JSTAT (2021) 114004.

$\left| \Psi_t \right\rangle = e^{-iHt} \left|$  $\right\rangle$**: large times**



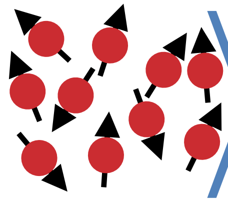
General setting

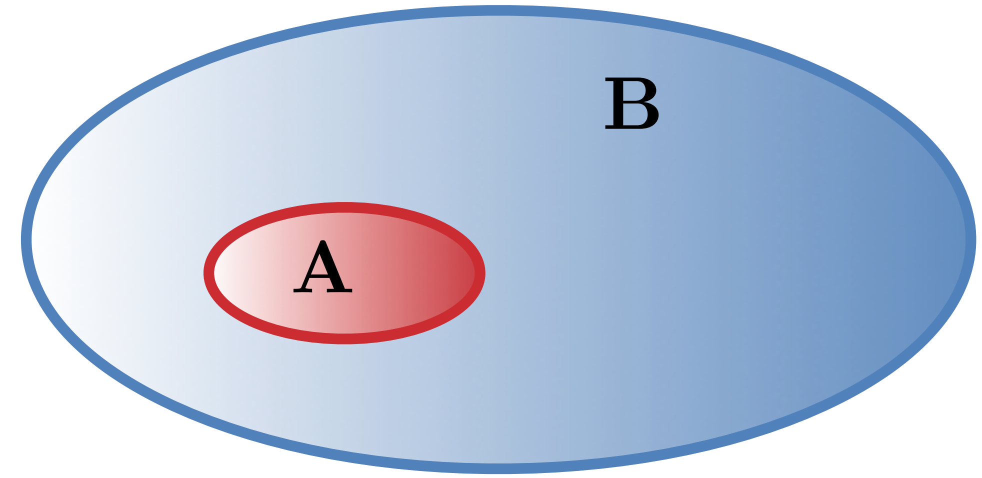
- Change variables $\{\beta_n(x,t)\} \mapsto \{q^{(n)}(x,t)\}$

$$\partial_t q^{(m)}(x,t) + \partial_x J^{(m)}[\{q^{(n)}(x,t)\}] = 0$$

Hydrodynamic Equations

↑ "Equations of state" (model dependent)

$\left| \Psi_t \right\rangle = e^{-iHt}$  : **large times**



General setting

- Change variables $\quad \{\beta_n(x,t)\} \mapsto \{q^{(n)}(x,t)\}$

$$\partial_t q^{(m)}(x,t) + \partial_x J^{(m)}[\{q^{(n)}(x,t)\}] = 0$$

Hydrodynamic Equations

↑ "Equations of state" (model dependent)

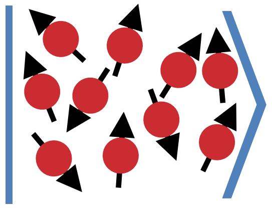
Example 1: $\{q^{(n)}(x,t)\} = e(x,t),\ n(x,t),\ p(x,t)$ + *Galilean Invariance*

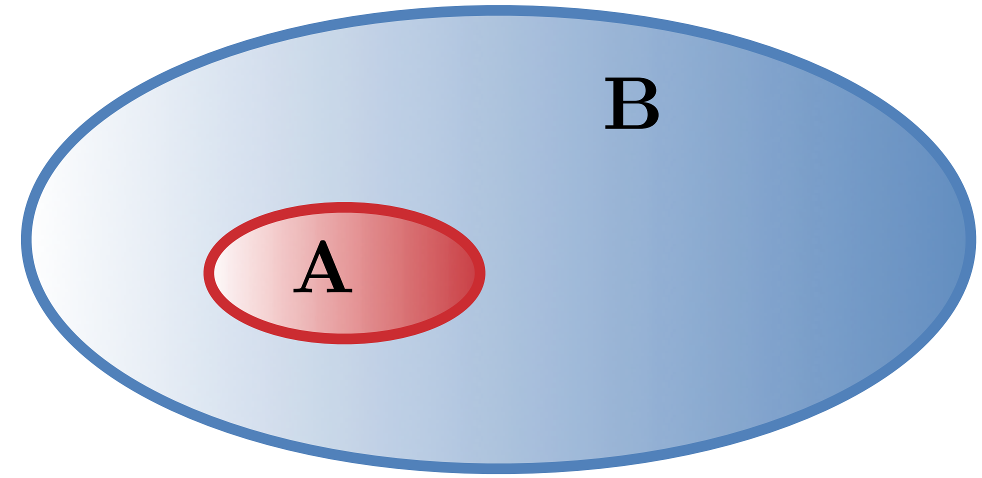
$$\partial_t n(x,t) + \partial_x (v(x,t) n(x,t)) = 0$$

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{1}{n(x,t)}\partial_x P[n(x,t)]$$

$$\partial_t e(x,t) + v(x,t)\partial_x e(x,t) = -\frac{P[n(x,t)]}{n(x,t)}\partial_x v(x,t)$$

Conventional Hydrodynamics

$\left|\Psi_t\right\rangle = e^{-iHt}$  : **large times**



General setting

- Change variables $\{\beta_n(x,t)\} \mapsto \{q^{(n)}(x,t)\}$

$$\partial_t q^{(m)}(x,t) + \partial_x J^{(m)}[\{q^{(n)}(x,t)\}] = 0$$

Hydrodynamic Equations

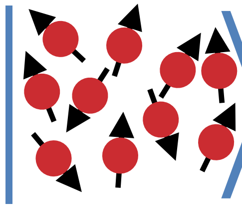
↑ "Equations of state" (model dependent)

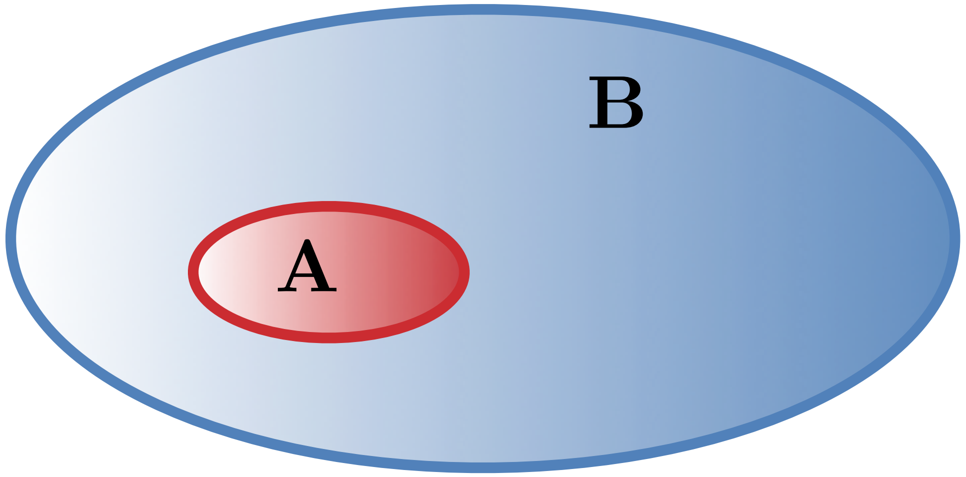
Example 2: $\{q^{(n)}(x,t)\}_{n=0,1,\ldots,\alpha L}$ ← **Volume** *"Integrable models"*

$$\hat{H}_{\mathrm{XXZ}} = J\hbar \sum_i \left(\hat{S}^x_i \hat{S}^x_{i+1} + \hat{S}^y_i \hat{S}^y_{i+1} + \Delta \hat{S}^z_i \hat{S}^z_{i+1}\right) + \hbar b \sum_i \hat{S}^z_i$$

$$\hat{H}_{\mathrm{FHM}} = \hbar t \sum_{\alpha=\uparrow,\downarrow} \sum_i \left(\hat{c}^\dagger_{\alpha,i}\hat{c}_{\alpha,i+1} + \mathrm{h.c.}\right) + \hbar U \sum_i \hat{n}_{i,\uparrow}\hat{n}_{i,\downarrow}$$

- Strongly interacting in 1d
- Stable quasiparticles excitations at finite energy density

$\left| \Psi_t \right\rangle = e^{-iHt}$ |  $\rangle$ **: large times**



General setting

- Change variables $\qquad \{\beta_n(x,t)\} \mapsto \{q^{(n)}(x,t)\}$

$$\partial_t q^{(m)}(x,t) + \partial_x J^{(m)}[\{q^{(n)}(x,t)\}] = 0 \qquad \textit{Hydrodynamic Equations}$$

↑ "Equations of state" (model dependent)

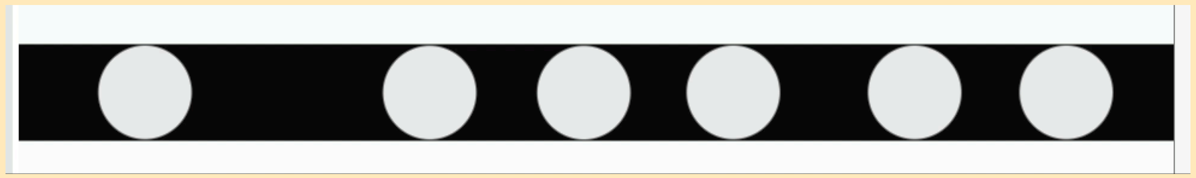
Example 2: $\quad \{q^{(n)}(x,t)\}_{n=0,1,\ldots,\infty}$ $\qquad$ *"Integrable models"*

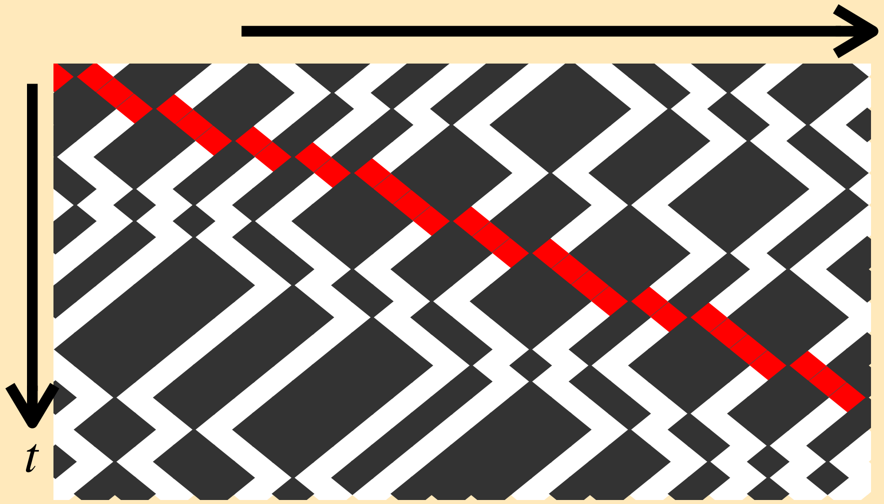
- Infinitely many equations treated by an appropriate "change of basis"

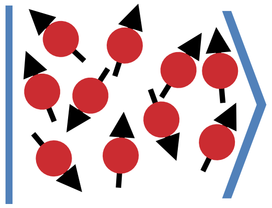
$$\{q^{(n)}(x,t)\} \longmapsto \rho(k,x,t)$$

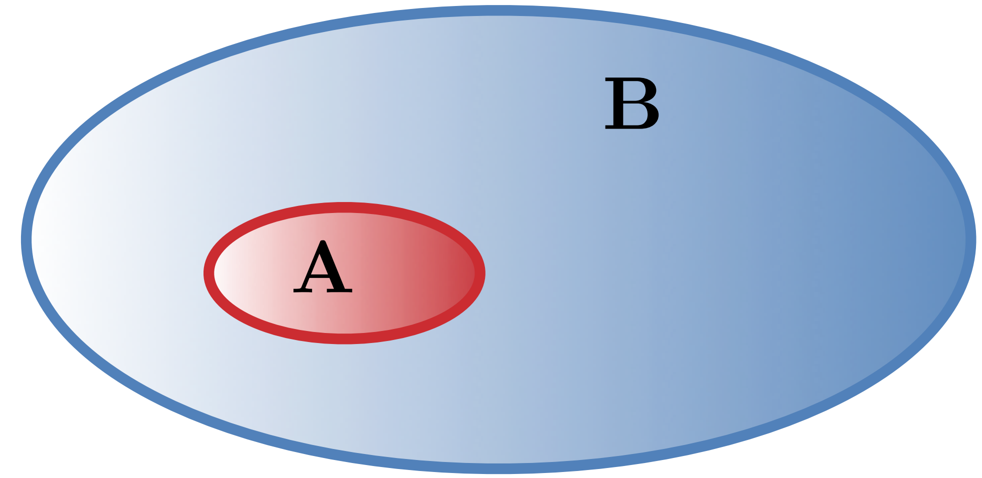
Density of quasiparticles with momentum k

- Equations of state known exactly





$|\Psi_t\rangle = e^{-iHt}\,|$  $\rangle$ **: large times**



General setting

- Change variables $\{\beta_n(x,t)\} \mapsto \{q^{(n)}(x,t)\}$

$$\partial_t q^{(m)}(x,t) + \partial_x J^{(m)}[\{q^{(n)}(x,t)\}] = 0$$

Hydrodynamic Equations

↑ "Equations of state" (model dependent)

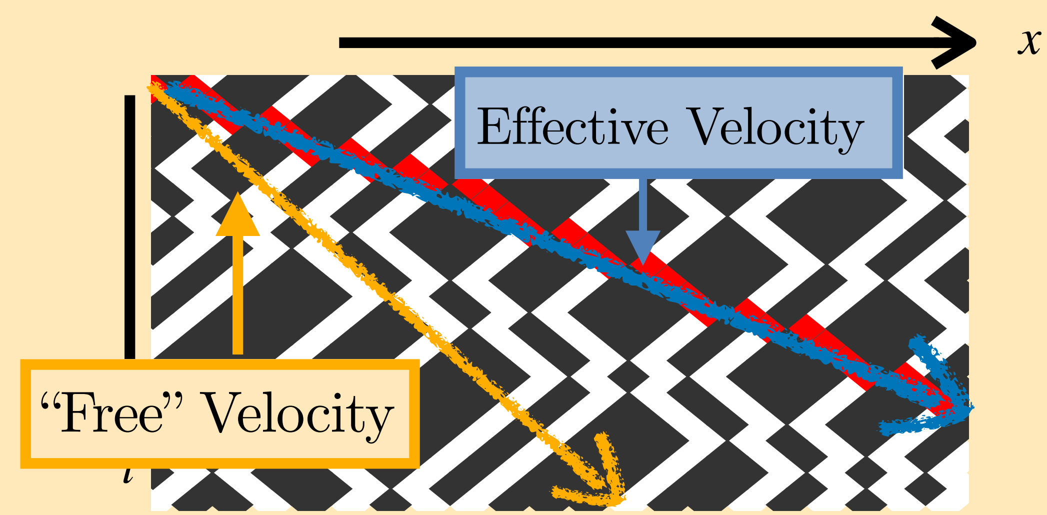
Example 2: $\{q^{(n)}(x,t)\}_{n=0,1,\dots,\infty}$ *"Integrable models"*

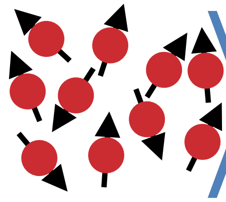
- Infinitely many equations treated by an appropriate "change of basis"

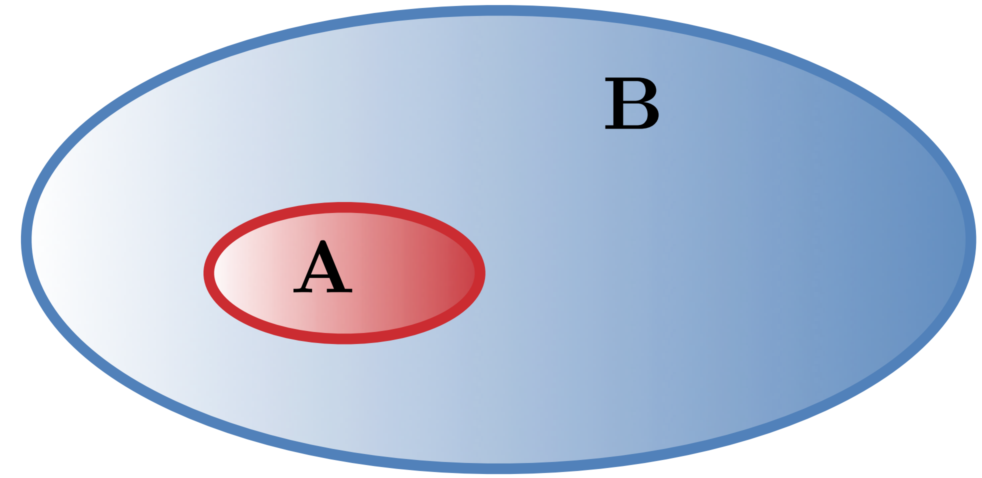
$$\{q^{(n)}(x,t)\} \longmapsto \rho(k,x,t)$$

Density of quasiparticles with momentum k

- Equations of state known exactly



$$\left|\Psi_t\right\rangle = e^{-iHt}\left|\right.$$  $$\left.\right\rangle$$ **: large times**



General setting

- Change variables $\{\beta_n(x,t)\} \mapsto \{q^{(n)}(x,t)\}$

$$\partial_t q^{(m)}(x,t) + \partial_x J^{(m)}[\{q^{(n)}(x,t)\}] = 0$$

Hydrodynamic Equations

↑ "Equations of state" (model dependent)

Example 2: $\{q^{(n)}(x,t)\}_{n=0,1,\ldots,\infty}$ *"Integrable models"*

- Infinitely many equations treated by an appropriate "change of basis"

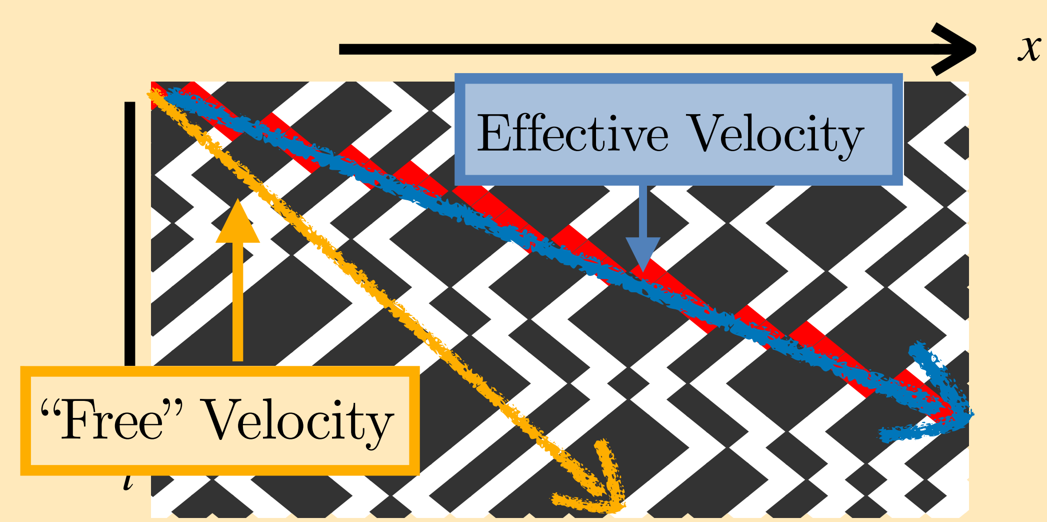
$$\{q^{(n)}(x,t)\} \longmapsto \rho(k,x,t)$$

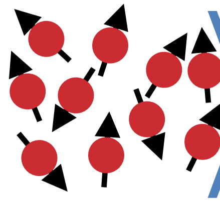
Density of quasiparticles with momentum k

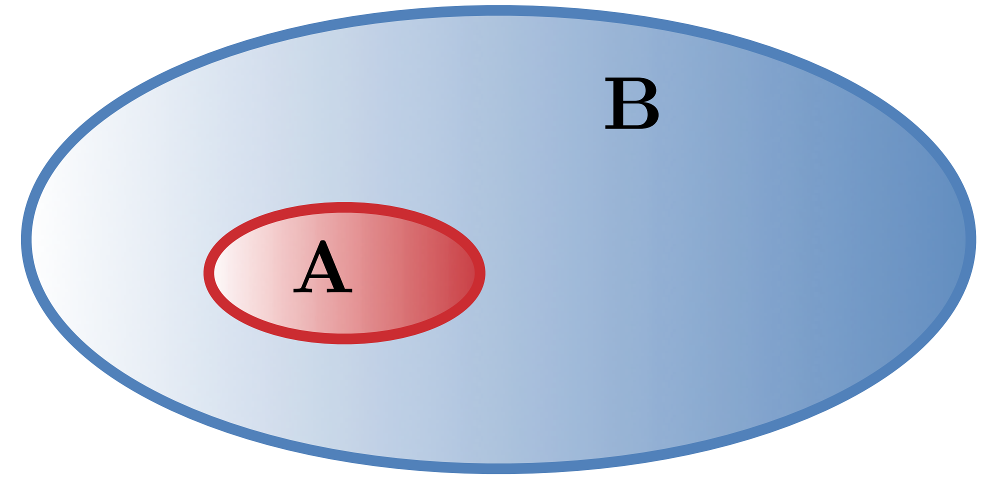
- Equations of state known exactly

$$v(k) = k + \int \mathrm{d}q\, K(k-q)(v(q) - v(k)\rho(q))$$

$$J_\rho = v(k)\rho(k)$$



$\left|\Psi_t\right\rangle = e^{-iHt}\Big|$  $\Big\rangle$ **: large times**



General setting

- Change variables $\{\beta_n(x,t)\} \mapsto \{q^{(n)}(x,t)\}$

$$\partial_t q^{(m)}(x,t) + \partial_x J^{(m)}[\{q^{(n)}(x,t)\}] = 0$$

Hydrodynamic Equations

↑ "Equations of state" (model dependent)

Example 2: $\{q^{(n)}(x,t)\}_{n=0,1,\dots,\infty}$ *"Integrable models"*

- Infinitely many equations treated by an appropriate "change of basis"

$$\{q^{(n)}(x,t)\} \longmapsto \rho(k,x,t)$$

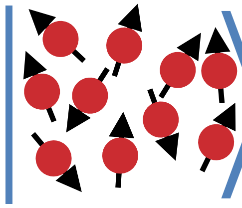
Density of quasiparticles with momentum k

- Equations of state known exactly

$$\partial_t \rho(k,x,t) + \partial_x(v(k,x,t)\rho(k,x,t)) = 0$$

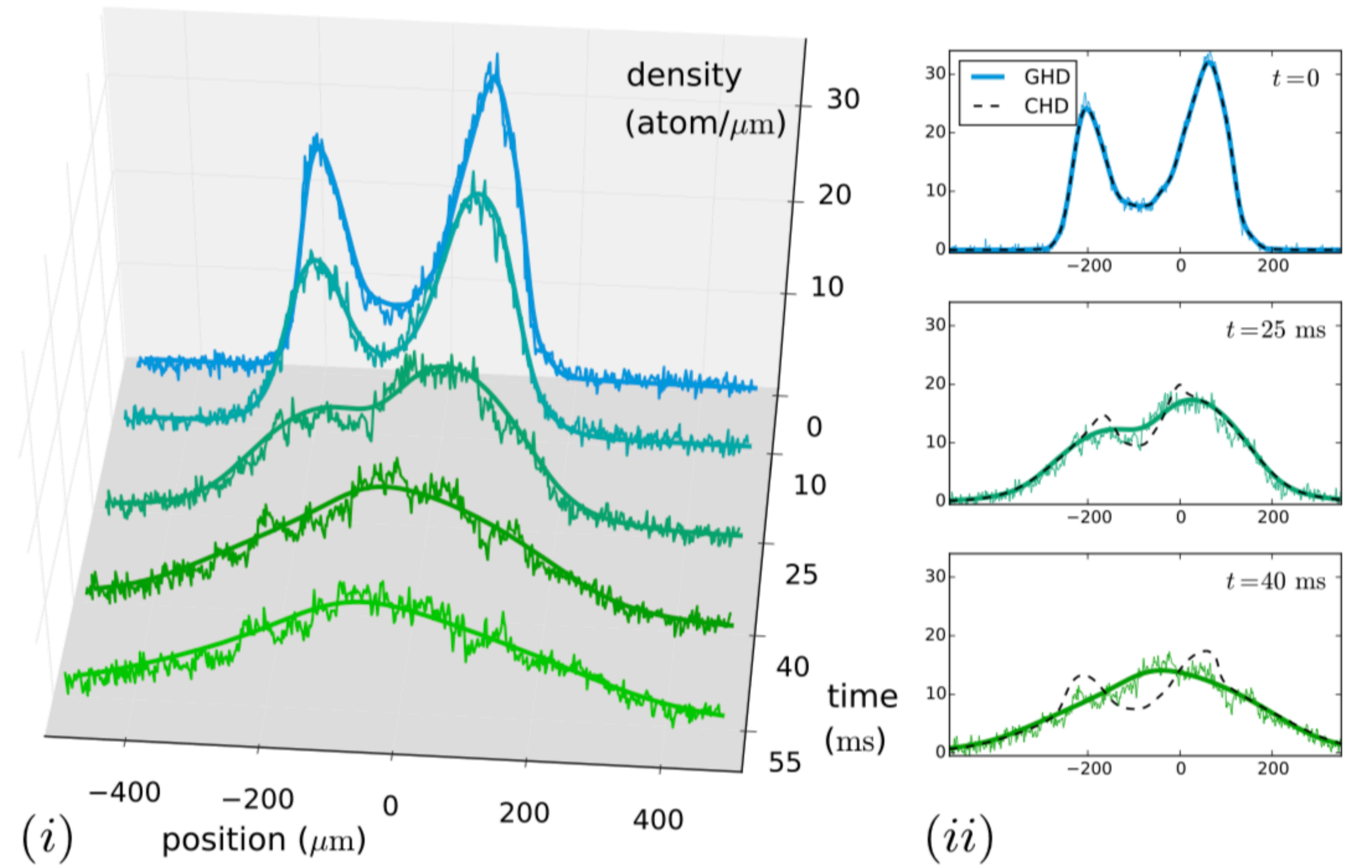
Generalized Hydrodynamics

BB et al. PRL **117**, 207201 (2016);
Castro-Alvaredo et al. PRX **6**, 041065 (2016).

$\left|\Psi_t\right\rangle = e^{-iHt}$  : large times

- *Describes 1d cold-atom experiments*!

Schemmer et al. PRL **122**, 090601 (2019).



Example 2: $\{q^{(n)}(x,t)\}_{n=0,1,\ldots,\infty}$ *"Integrable models"*

- Infinitely many equations treated by an appropriate "change of basis"

$$\{q^{(n)}(x,t)\} \longmapsto \rho(k,x,t)$$

Density of quasiparticles with momentum k

- Equations of state known exactly

$$\partial_t \rho(k,x,t) + \partial_x(v(k,x,t)\rho(k,x,t)) = 0$$

Generalized Hydrodynamics

BB et al. PRL **117**, 207201 (2016);
Castro-Alvaredo et al. PRX **6**, 041065 (2016).

- Infinitely many equations treated by an appropriate "change of basis"

$$\{q^{(n)}(x,t)\} \quad \longmapsto \quad \rho(k,x,t)$$

Can we understand this mapping "microscopically"?

- Express $\rho(k,x,t)$ in terms of the **operators of the theory**

- Understand what fails for non-integrable systems

- Infinitely many equations treated by an appropriate "change of basis"

$$\{q^{(n)}(x,t)\} \quad \longmapsto \quad \rho(k,x,t)$$

Can we understand this mapping "microscopically"?

- Express $\rho(k,x,t)$ in terms of the **operators of the theory**

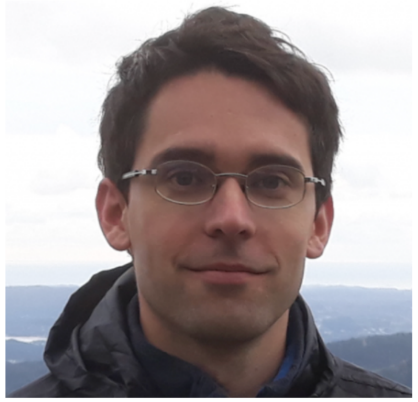
- Understand what fails for non-integrable systems

Fabian Essler

Oxford

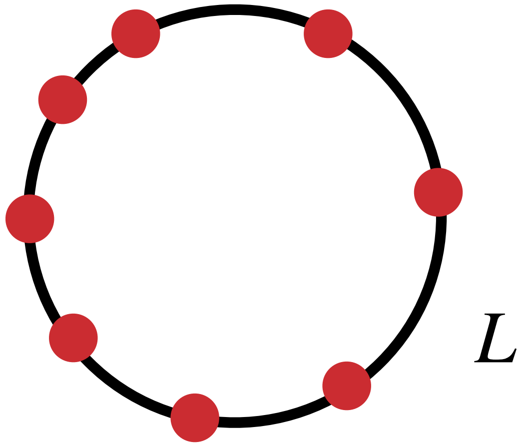
BB, Essler, and Granet, arXiv:2201.10549 (2022)

Etienne Granet



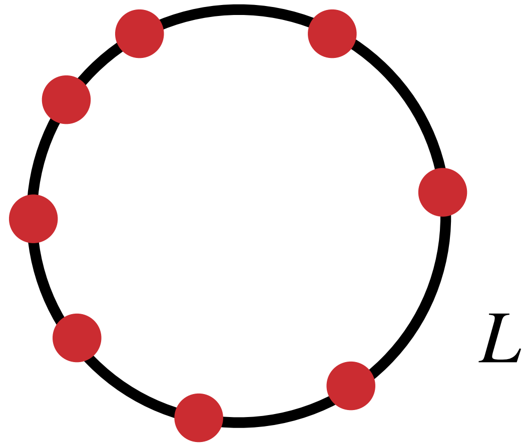
Chicago

(II) Weakly interacting fermionic quantum gases in 1d and BBGKY hierarchy



$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

$$\beta \ll 1 \qquad\qquad \{\psi^\dagger(x), \psi(y)\} = \delta(x-y)$$



$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

$$\beta \ll 1 \qquad\qquad \{\psi^\dagger(x), \psi(y)\} = \delta(x-y)$$

- Contains integrable points

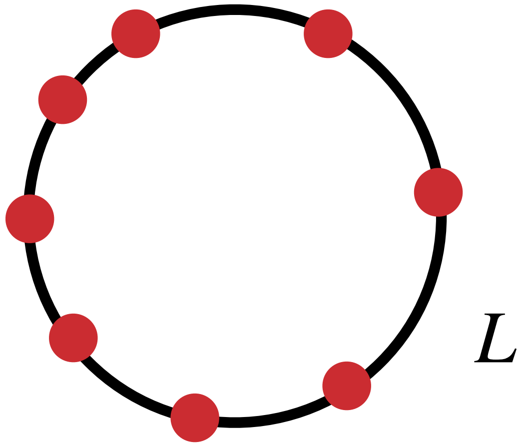
  - *Calogero-Sutherland* potentials of the form $V(x) = \mathscr{P}(x, L, \omega)$ ← *Weierstrass Elliptic Function*

    e.g. $V(x) \propto \frac{1}{x^2}, \frac{1}{\sin^2\left(\frac{\pi}{L}x\right)}, \frac{1}{\sinh^2\left(\frac{x}{\alpha}\right)}, \ldots$

  - *Cheon-Shigehara* : Fermionic formulation of Lieb-Liniger

    Cheon and Shigehara, PRL **82**, 2536 (1999)

    - Fermions are *weakly coupled* for *strongly coupled* bosons



$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

$$\beta \ll 1 \qquad\qquad \{\psi^\dagger(x), \psi(y)\} = \delta(x-y)$$

- Contains integrable points
  - *Calogero-Sutherland* potentials of the form $V(x) = \mathcal{P}(x, L, \omega)$ ← *Weierstrass Elliptic Function*

    e.g. $\quad V(x) \propto \dfrac{1}{x^2}, \dfrac{1}{\sin^2\left(\frac{\pi}{L}x\right)}, \dfrac{1}{\sinh^2\left(\frac{x}{\alpha}\right)}, \ldots$

  - *Cheon-Shigehara* : Fermionic formulation of Lieb-Liniger

    Cheon and Shigehara, PRL **82**, 2536 (1999)

    - Fermions are *weakly coupled* for *strongly coupled* bosons
    - 2nd quantized potential for fermions found recently

$$V_\epsilon(x) = \frac{1}{\epsilon^2} \frac{\sigma''(x/\epsilon)}{x + \beta\sigma(x/\epsilon)}$$

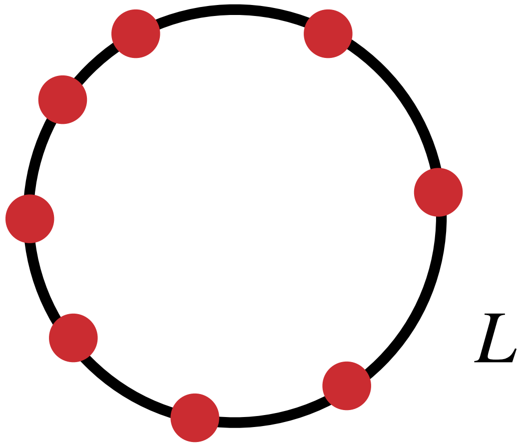
Granet, BB, and Essler, PRL **128**, 021604 (2022)

ϵ regulator

$\sigma(x)$ smooth, s.t.

$\sigma'(0) > 0 \qquad \sigma'(x) \geq 0$

$\lim_{x\to\infty} \sigma(x) = 1 \qquad \lim_{x\to\infty} x^2\sigma''(x) = 0$

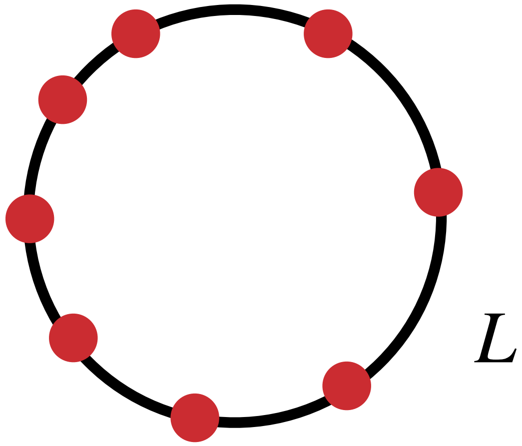


$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

$$\beta \ll 1 \qquad \{\psi^\dagger(x), \psi(y)\} = \delta(x-y)$$

- Contains integrable points

  - *Calogero-Sutherland* potentials of the form $V(x) = \mathscr{P}(x, L, \omega)$
  - *Cheon-Shigehara*

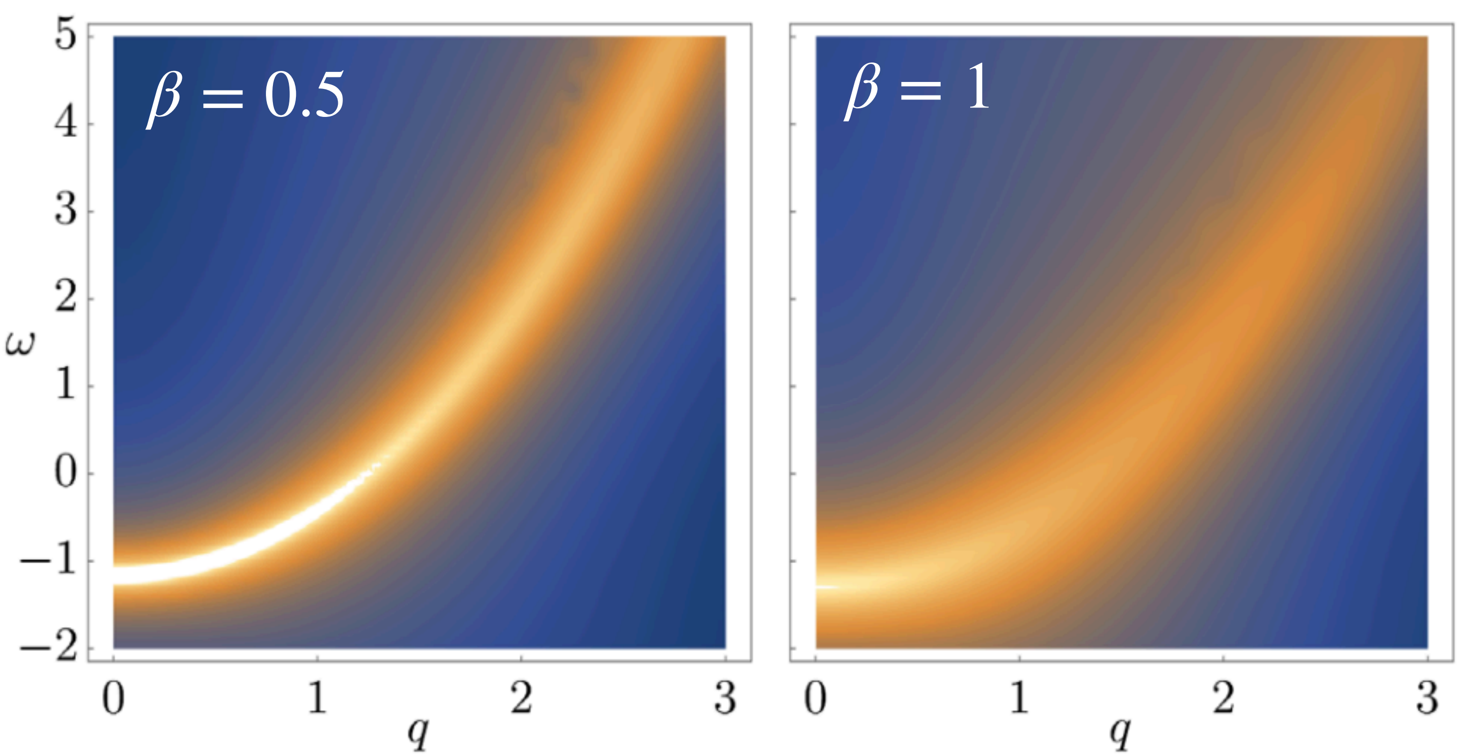


$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

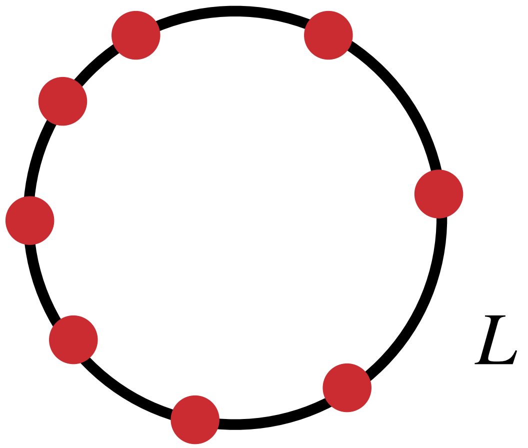
$\beta \ll 1$ $\qquad$ $\{\psi^\dagger(x), \psi(y)\} = \delta(x-y)$

- Contains integrable points
  - *Calogero-Sutherland* potentials of the form $V(x) = \mathscr{P}(x, L, \omega)$
  - *Cheon-Shigehara*

- No bound states: quasiparticles smoothly connected to free fermions



$\Rightarrow$ No abrupt changes in the physics for $\beta > 0$

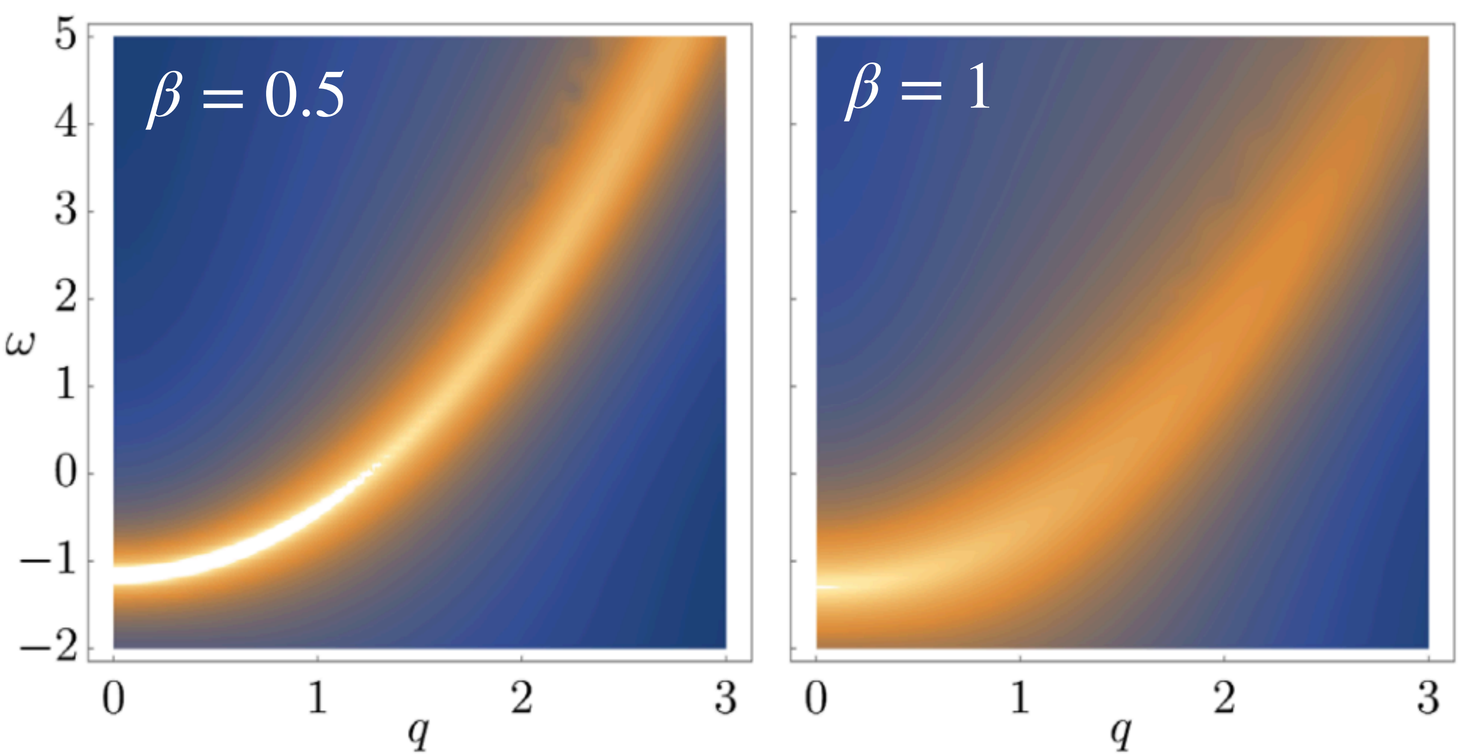


Minimal Setting

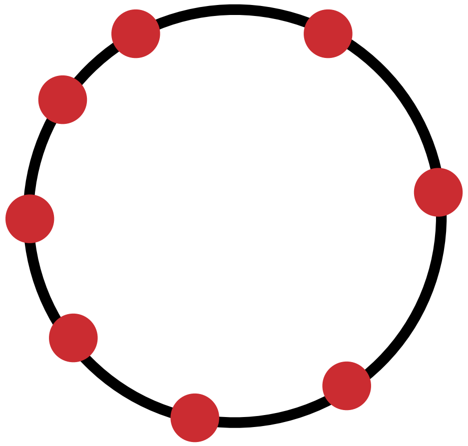
$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

$$\beta \ll 1 \qquad \{\psi^\dagger(x), \psi(y)\} = \delta(x-y)$$

- Contains integrable points
  - *Calogero-Sutherland* potentials of the form $V(x) = \mathscr{P}(x, L, \omega)$
  - *Cheon-Shigehara*
- No bound states: quasiparticles smoothly connected to free fermions



$\Rightarrow$ No abrupt changes in the physics for $\beta > 0$



Minimal Setting

$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

Standard description in terms of "n-particle density matrices"

$$\rho_n(\mathbf{x},\mathbf{y};t) = \mathrm{Tr}\left[\rho(t)\psi^\dagger(x_1)\ldots\psi^\dagger(x_n)\psi\,(y_n)\ldots\psi\,(y_1)\right]$$

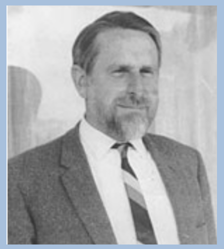
Fulfilling

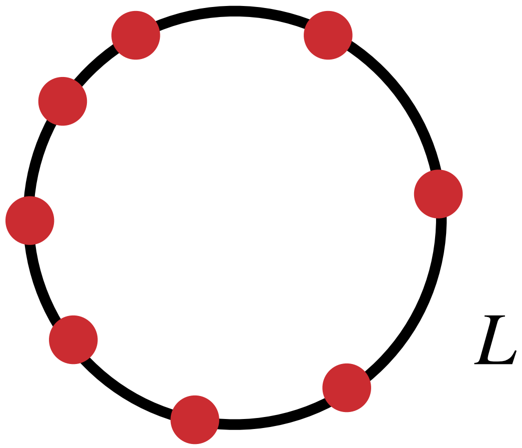
$$-\sum_{j=1}^{n}\partial_{x_j}^2 + 2\beta\sum_{i>j}^{n} V(x_i - x_j)$$

$$i\partial_t\rho_n(\mathbf{x},\mathbf{y};t) - (H_{\mathbf{x}}^{(n)} - H_{\mathbf{y}}^{(n)})\rho_n(\mathbf{x},\mathbf{y};t) = \sum_{j=1}^{n}\int dw[V(x_j - w) - V(y_j - w)]\rho_{n+1}(\mathbf{x},w,\mathbf{y},w;t)$$

BBGKY hierarchy







Minimal Setting

$$H = -\int_0^L \mathrm{d}x\, \psi(x)^\dagger \partial_x^2 \psi(x) + \beta \int_0^L V(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

Standard description in terms of "n-particle density matrices"

$$\rho_n(\mathbf{x},\mathbf{y};t) = \mathrm{Tr}\left[\rho(t)\psi^\dagger(x_1)\ldots\psi^\dagger(x_n)\psi\,(y_n)\ldots\psi\,(y_1)\right]$$

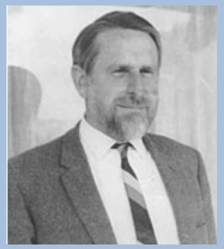
Fulfilling

$$-\sum_{j=1}^{n}\partial_{x_j}^2 + 2\beta\sum_{i>j}^{n} V(x_i - x_j)$$

$$i\partial_t\rho_n(\mathbf{x},\mathbf{y};t) - (H_{\mathbf{x}}^{(n)} - H_{\mathbf{y}}^{(n)})\rho_n(\mathbf{x},\mathbf{y};t) = \sum_{j=1}^{n}\int dw[V(x_j - w) - V(y_j - w)]\rho_{n+1}(\mathbf{x},w,\mathbf{y},w;t)$$

BBGKY hierarchy





Alternative "representations"

$$\rho_n(\mathbf{x},\mathbf{y};t) \quad\mapsto\quad \rho_n(\mathbf{q},\mathbf{k};t) = \int \mathbf{dxdy}\, e^{i(\mathbf{qx}-\mathbf{ky})}\rho_n(\mathbf{x},\mathbf{y};t)$$

"Momentum Space"

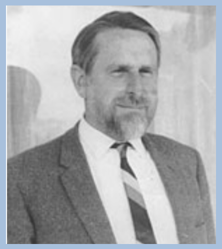
$$\rho_n(\mathbf{x},\mathbf{y};t) \quad\mapsto\quad \rho_n(\mathbf{k},\mathbf{x};t) = \int \mathbf{dz}\, e^{i\mathbf{zk}}\rho_n(\mathbf{x}-\mathbf{z}/2,\mathbf{x}+\mathbf{z}/2;t)$$

"Wigner"

$$i\partial_t \rho_n(\mathbf{x},\mathbf{y};t) - (H^{(n)}_{\mathbf{x}} - H^{(n)}_{\mathbf{y}})\rho_n(\mathbf{x},\mathbf{y};t) = \sum_{j=1}^{n} \int dw [V(x_j - w) - V(y_j - w)]\rho_{n+1}(\mathbf{x}, w, \mathbf{y}, w; t)$$

BBGKY hierarchy





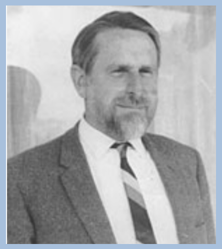
- Impractical for general β (infinitely many equations!)

- For small β can be truncated

$$i\partial_t \rho_n(\mathbf{x},\mathbf{y};t) - (H^{(n)}_{\mathbf{x}} - H^{(n)}_{\mathbf{y}})\rho_n(\mathbf{x},\mathbf{y};t) = \sum_{j=1}^{n}\int dw[V(x_j - w) - V(y_j - w)]\rho_{n+1}(\mathbf{x},w,\mathbf{y},w;t)$$

BBGKY hierarchy





- Impractical for general β (infinitely many equations!)
- For small β can be truncated

Example: *Quantum Boltzmann Equation*

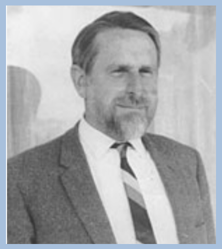
$$\rho_4(\mathbf{x},\mathbf{k};t) = G(\{\rho_2(x,k;t)\}) \qquad \lim_{\beta\to 0}\rho_2(x/\beta^2,k;t/\beta^2) = F(x,k;t)$$

"Collisional" derivative

$$\partial_t F(x,k_1;t) + k_1\partial_x F(x,k_1;t) \propto \int \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\,\delta(k_1+k_2-k_3-k_4)\delta(k_1^2+k_2^2-k_3^2-k_4^2)(\tilde{V}(k_1-k_3)-\tilde{V}(k_1-k_4))^2$$
$$\times (F_{k_1}F_{k_2}(1-F_{k_3})(1-F_{k_4}) - (1-F_{k_1})(1-F_{k_2})F_{k_3}F_{k_4})$$

$$i\partial_t \rho_n(\mathbf{x},\mathbf{y};t) - (H^{(n)}_{\mathbf{x}} - H^{(n)}_{\mathbf{y}})\rho_n(\mathbf{x},\mathbf{y};t) = \sum_{j=1}^{n} \int dw[V(x_j - w) - V(y_j - w)]\rho_{n+1}(\mathbf{x},w,\mathbf{y},w;t)$$

BBGKY hierarchy

- Impractical for general β (infinitely many equations!)

- For small β can be truncated

Example: *Quantum Boltzmann Equation*

$$\rho_4(\mathbf{x},\mathbf{k};t) = G(\{\rho_2(x,k;t)\}) \qquad \lim_{\beta\to 0} \rho_2(x/\beta^2, k; t/\beta^2) = F(x,k;t)$$

"Collisional" derivative

$$\partial_t F(x,k_1;t) + k_1 \partial_x F(x,k_1;t) \propto \int \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 + k_2 - k_3 - k_4)\delta(k_1^2 + k_2^2 - k_3^2 - k_4^2)(\tilde{V}(k_1 - k_3) - \tilde{V}(k_1 - k_4))^2$$
$$\times (F_{k_1}F_{k_2}(1 - F_{k_3})(1 - F_{k_4}) - (1 - F_{k_1})(1 - F_{k_2})F_{k_3}F_{k_4})$$

Recall GHD equations

$$\partial_t \rho(k,x,t) + \partial_x(v(k,x,t)\rho(k,x,t)) = 0$$

Non collisional!

$$i\partial_t \rho_n(\mathbf{x},\mathbf{y};t) - (H^{(n)}_{\mathbf{x}} - H^{(n)}_{\mathbf{y}})\rho_n(\mathbf{x},\mathbf{y};t) = \sum_{j=1}^{n} \int dw [V(x_j - w) - V(y_j - w)]\rho_{n+1}(\mathbf{x}, w, \mathbf{y}, w; t)$$

BBGKY hierarchy

- Impractical for general β (infinitely many equations!)

- For small β can be truncated

Example: *Quantum Boltzmann Equation*

$$\rho_4(\mathbf{x},\mathbf{k};t) = G(\{\rho_2(x,k;t)\}) \qquad \lim_{\beta\to 0} \rho_2(x/\beta^2, k; t/\beta^2) = F(x,k;t)$$

"Collisional" derivative

$$\partial_t F(x,k_1;t) + k_1 \partial_x F(x,k_1;t) \propto \int \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 + k_2 - k_3 - k_4)\delta(k_1^2 + k_2^2 - k_3^2 - k_4^2)(\tilde{V}(k_1 - k_3) - \tilde{V}(k_1 - k_4))^2$$
$$\times (F_{k_1}F_{k_2}(1 - F_{k_3})(1 - F_{k_4}) - (1 - F_{k_1})(1 - F_{k_2})F_{k_3}F_{k_4})$$

Recall GHD equations

$$\partial_t \rho(k,x,t) + \partial_x (v(k,x,t)\rho(k,x,t)) = 0$$

Non collisional!

$$i\partial_t \rho_n(\mathbf{x},\mathbf{y};t) - (H^{(n)}_{\mathbf{x}} - H^{(n)}_{\mathbf{y}})\rho_n(\mathbf{x},\mathbf{y};t) = \sum_{j=1}^{n} \int dw [V(x_j - w) - V(y_j - w)]\rho_{n+1}(\mathbf{x}, w, \mathbf{y}, w; t)$$

BBGKY hierarchy

- Impractical for general β (infinitely many equations!)
- For small β can be truncated

Example: *Quantum Boltzmann Equation*

$$\rho_4(\mathbf{x},\mathbf{k};t) = G(\{\rho_2(x,k;t)\}) \qquad \lim_{\beta\to 0} \rho_2(x/\beta^2, k; t/\beta^2) = F(x,k;t)$$

"Collisional" derivative

$$\partial_t F(x,k_1;t) + k_1 \partial_x F(x,k_1;t) \propto \int \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 + k_2 - k_3 - k_4)\delta(k_1^2 + k_2^2 - k_3^2 - k_4^2)(\tilde{V}(k_1 - k_3) - \tilde{V}(k_1 - k_4))^2 \times (F_{k_1}F_{k_2}(1 - F_{k_3})(1 - F_{k_4}) - (1 - F_{k_1})(1 - F_{k_2})F_{k_3}F_{k_4})$$

Recall GHD equations

$$\partial_t \rho(k,x,t) + \partial_x(v(k,x,t)\rho(k,x,t)) = 0$$

Non collisional!

How can we match the two?

(III) Matching the two descriptions

Strategy:

1. Construct the charges of H in perturbation theory

2. Build the "operatorial density of quasiparticles" using the them

1. Construct the charges of H in perturbation theory

Define:

$$Q = \sum_{m=0}^{\infty} Q_m \qquad H = H_0 + H_1 \qquad O_m \text{ order } m \text{ in } \beta$$

A. Build charges *recursively* in momentum space using

$$[H_1, Q_{m-1}] = -[H_0, Q_m] \qquad (1)$$

$$Q_{f;0} = \sum_p f(p)\psi_p^\dagger \psi_p \qquad f(p) \text{ smooth function}$$

B. Investigate locality properties mapping back to real space

1. Construct the charges of H in perturbation theory

Define:

$$Q = \sum_{m=0}^{\infty} Q_m \qquad H = H_0 + H_1 \qquad O_m \text{ order } m \text{ in } \beta$$

A. Build charges *recursively* in momentum space using

$$[H_1, Q_{m-1}] = -[H_0, Q_m] \qquad (1)$$

$$Q_{f;0} = \sum_p f(p)\psi_p^\dagger \psi_p \qquad f(p) \text{ smooth function}$$

B. Investigate locality properties mapping back to real space

Observation 1:

$$Q_{f;0} = \int \mathrm{d}x\mathrm{d}y\, \tilde{f}(x-y)\psi^\dagger(x)\psi(y)$$

quasi-local for any $f(p)$

1. Construct the charges of H in perturbation theory

Define:

$$Q = \sum_{m=0}^{\infty} Q_m \qquad H = H_0 + H_1 \qquad O_m \text{ order } m \text{ in } \beta$$

A. Build charges *recursively* in momentum space using

$$[H_1, Q_{m-1}] = -[H_0, Q_m] \qquad (1)$$

$$Q_{f;0} = \sum_p f(p)\psi_p^\dagger \psi_p \qquad f(p) \text{ smooth function}$$

B. Investigate locality properties mapping back to real space

Observation 1:

$$Q_{f;0} = \int \mathrm{d}x\mathrm{d}y\, \tilde{f}(x-y)\psi^\dagger(x)\psi(y)$$

quasi-local for any $f(p)$

Observation 2:

Eq. (1) is invariant under

$$Q_m \mapsto Q_m + D_m \qquad [D_m, H_0] = 0$$

we do not add these terms: **minimality**

1. Construct the charges of H in perturbation theory

A. Build charges *recursively* in momentum space using

$$[H_1, Q_{m-1}] = -[H_0, Q_m] \qquad (1)$$

$$Q_{f;0} = \sum_p f(p)\psi_p^\dagger \psi_p \qquad f(p) \text{ smooth function}$$

B. Investigate locality properties mapping back to real space

- At first order (1) always gives quasi-local solutions

$$Q_{f;1} = \frac{1}{L}\sum_{\boldsymbol{k}} g_{f;1}^{(4)}(k_1, k_2, k_3)\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3} \psi_{k_1+k_2-k_3} = \int \mathrm{d}x_1 \cdots \mathrm{d}x_4 \, \tilde{g}_{f;1}^{(4)}(x_1 - x_4, x_2 - x_4, x_3 - x_4)\, \psi^\dagger(x_1) \cdots \psi(x_4)$$

$g_{f;1}^{(4)}(k_1, k_2, k_3)$ non-singular $\Rightarrow$ $Q_{f;1}$ quasi-local

1. Construct the charges of H in perturbation theory

A. Build charges *recursively* in momentum space using

$$[H_1, Q_{m-1}] = -[H_0, Q_m] \qquad (1)$$

$$Q_{f;0} = \sum_p f(p)\psi_p^\dagger \psi_p \qquad f(p) \text{ smooth function}$$

B. Investigate locality properties mapping back to real space

- At first order (1) always gives quasi-local solutions

$$Q_{f;1} = \frac{1}{L}\sum_{\boldsymbol{k}} g_{f;1}^{(4)}(k_1,k_2,k_3)\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}\psi_{k_1+k_2-k_3} = \int \mathrm{d}x_1 \cdots \mathrm{d}x_4\, \tilde{g}_{f;1}^{(4)}(x_1-x_4, x_2-x_4, x_3-x_4)\, \psi^\dagger(x_1)\cdots\psi(x_4)$$

$g_{f;1}^{(4)}(k_1,k_2,k_3)$ non-singular $\Rightarrow$ $Q_{f;1}$ quasi-local

Explicitly:

$$g_{f;1}^{(4)}(k_1,k_2,k_3) = (V(k_1-k_3) - V(k_2-k_3))\frac{f(k_1)+f(k_2)-f(k_3)-f(k_1+k_2-k_3)}{(k_1+k_2-k_3)^2+k_3^2-k_2^2-k_1^2}$$

1. Construct the charges of H in perturbation theory

A. Build charges *recursively* in momentum space using

$$[H_1, Q_{m-1}] = -[H_0, Q_m] \qquad (1)$$

$$Q_{f;0} = \sum_p f(p)\psi_p^\dagger \psi_p \qquad f(p) \text{ smooth function}$$

B. Investigate locality properties mapping back to real space

- At first order (1) always gives quasi-local solutions

$$Q_{f;1} = \frac{1}{L}\sum_{\boldsymbol{k}} g_{f;1}^{(4)}(k_1,k_2,k_3)\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}\psi_{k_1+k_2-k_3} = \int \mathrm{d}x_1 \cdots \mathrm{d}x_4\, \tilde{g}_{f;1}^{(4)}(x_1-x_4, x_2-x_4, x_3-x_4)\, \psi^\dagger(x_1)\cdots\psi(x_4)$$

$g_{f;1}^{(4)}(k_1,k_2,k_3)$ non-singular $\Rightarrow$ $Q_{f;1}$ quasi-local

- At higher order (1) admits quasi-local solutions <u>only if the potential fulfils certain conditions</u>

1. Construct the charges of H in perturbation theory

A. Build charges *recursively* in momentum space using

$$[H_1, Q_{m-1}] = -[H_0, Q_m] \qquad (1)$$

$$Q_{f;0} = \sum_p f(p)\psi^\dagger_p \psi_p \qquad f(p) \text{ smooth function}$$

B. Investigate locality properties mapping back to real space

- At first order (1) always gives quasi-local solutions

$$Q_{f;1} = \frac{1}{L}\sum_{\boldsymbol{k}} g^{(4)}_{f;1}(k_1, k_2, k_3)\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi_{k_3}\psi_{k_1+k_2-k_3} = \int \mathrm{d}x_1 \cdots \mathrm{d}x_4\, \tilde{g}^{(4)}_{f;1}(x_1 - x_4, x_2 - x_4, x_3 - x_4)\, \psi^\dagger(x_1)\cdots\psi(x_4)$$

$g^{(4)}_{f;1}(k_1, k_2, k_3)$ non-singular $\Rightarrow$ $Q_{f;1}$ quasi-local

- At higher order (1) admits quasi-local solutions <u>only if the potential fulfils certain conditions</u>

What are these conditions?

What are these conditions?

Second order condition

singular for generic V and f

$$Q_{f;2} = \frac{1}{L}\sum_{\boldsymbol{k}} g^{(4)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi_{k_3}\psi_{k_1+k_2-k_3} + \frac{1}{L^2}\sum_{\boldsymbol{k}} g^{(6)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi^\dagger_{k_3}\psi_{k_4}\psi_{k_5}\psi_{k_1+k_2+k_3-k_4-k_5}$$

Condition 1.

The *second order coefficient is regular only if*

$$\mathscr{A}_{k_1k_2k_3}\mathscr{A}_{k_4k_5k_6}\left[\frac{V(k_4-k_1)V(k_3-k_5)}{(k_1-k_4)(k_2-k_4)}\right] = 0$$

$$k_1^2+k_2^2+k_3^2 = k_4^2+k_5^2+k_6^2$$
$$k_1+k_2+k_3 = k_4+k_5+k_6$$
$$\{k_1,k_2,k_3\} \neq \{k_4,k_5,k_6\}$$

What are these conditions?

Second order condition

$$Q_{f;2} = \frac{1}{L}\sum_{\boldsymbol{k}} g^{(4)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi_{k_3}\psi_{k_1+k_2-k_3} + \frac{1}{L^2}\sum_{\boldsymbol{k}} g^{(6)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi^\dagger_{k_3}\psi_{k_4}\psi_{k_5}\psi_{k_1+k_2+k_3-k_4-k_5}$$

Condition 1.

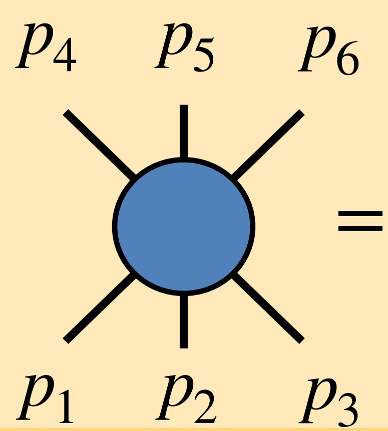
The *second order coefficient is regular only if*

$$\mathscr{A}_{k_1k_2k_3}\mathscr{A}_{k_4k_5k_6}\left[\frac{V(k_4-k_1)V(k_3-k_5)}{(k_1-k_4)(k_2-k_4)}\right] = 0$$

$$k_1^2+k_2^2+k_3^2 = k_4^2+k_5^2+k_6^2$$
$$k_1+k_2+k_3 = k_4+k_5+k_6$$
$$\{k_1,k_2,k_3\} \neq \{k_4,k_5,k_6\}$$

Observation.

Equivalent to the vanishing of the **inelastic component** of the **three-particle S-matrix at second order**



$= 0$

$$\{p_1,p_2,p_3\} \neq \{p_4,p_5,p_6\}$$

BB, Essler, and Granet, in preparation

What are these conditions?

Second order condition

$$Q_{f;2} = \frac{1}{L}\sum_{\boldsymbol{k}} g^{(4)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi_{k_3}\psi_{k_1+k_2-k_3} + \frac{1}{L^2}\sum_{\boldsymbol{k}} g^{(6)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi^\dagger_{k_3}\psi_{k_4}\psi_{k_5}\psi_{k_1+k_2+k_3-k_4-k_5}$$

Condition 1.

The *second order coefficient is regular only if*

$$\mathscr{A}_{k_1k_2k_3}\mathscr{A}_{k_4k_5k_6}\left[\frac{V(k_4-k_1)V(k_3-k_5)}{(k_1-k_4)(k_2-k_4)}\right] = 0$$

$$k_1^2+k_2^2+k_3^2 = k_4^2+k_5^2+k_6^2$$
$$k_1+k_2+k_3 = k_4+k_5+k_6$$
$$\{k_1,k_2,k_3\} \neq \{k_4,k_5,k_6\}$$

Theorem.

The only potentials fulfilling Condition 1 and admitting a power-series expansion around 0 are

$$V(k) = \alpha\left(1-\sqrt{\gamma}k\coth\sqrt{\gamma}k\right), \quad \alpha,\gamma\in\mathbb{R}$$

What are these conditions?

Second order condition

$$Q_{f;2} = \frac{1}{L}\sum_{\boldsymbol{k}} g^{(4)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi_{k_3}\psi_{k_1+k_2-k_3} + \frac{1}{L^2}\sum_{\boldsymbol{k}} g^{(6)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi^\dagger_{k_3}\psi_{k_4}\psi_{k_5}\psi_{k_1+k_2+k_3-k_4-k_5}$$

Condition 1.

The *second order coefficient is regular only if*

$$\mathscr{A}_{k_1k_2k_3}\mathscr{A}_{k_4k_5k_6}\left[\frac{V(k_4-k_1)V(k_3-k_5)}{(k_1-k_4)(k_2-k_4)}\right] = 0$$

$$k_1^2+k_2^2+k_3^2 = k_4^2+k_5^2+k_6^2$$
$$k_1+k_2+k_3 = k_4+k_5+k_6$$
$$\{k_1,k_2,k_3\} \neq \{k_4,k_5,k_6\}$$

Theorem.

The only potentials fulfilling Condition 1 and admitting a power-series expansion around 0 are

$$V(k) = \alpha\left(1-\sqrt{\gamma}k\coth\sqrt{\gamma}k\right), \quad \alpha,\gamma\in\mathbb{R}$$

Covers *all* and *only* integrable instances of the theory

What are these conditions?

Second order condition

$$Q_{f;2} = \frac{1}{L}\sum_{\boldsymbol{k}} g^{(4)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi_{k_3}\psi_{k_1+k_2-k_3} + \frac{1}{L^2}\sum_{\boldsymbol{k}} g^{(6)}_{f;2}(\boldsymbol{k})\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi^\dagger_{k_3}\psi_{k_4}\psi_{k_5}\psi_{k_1+k_2+k_3-k_4-k_5}$$

Condition 1.

The *second order coefficient is regular only if*

$$\mathscr{A}_{k_1k_2k_3}\mathscr{A}_{k_4k_5k_6}\left[\frac{V(k_4-k_1)V(k_3-k_5)}{(k_1-k_4)(k_2-k_4)}\right] = 0$$

$$k_1^2+k_2^2+k_3^2 = k_4^2+k_5^2+k_6^2$$
$$k_1+k_2+k_3 = k_4+k_5+k_6$$
$$\{k_1,k_2,k_3\} \neq \{k_4,k_5,k_6\}$$

Theorem.

The only potentials fulfilling Condition 1 and admitting a power-series expansion around 0 are

$$V(k) = \alpha\left(1-\sqrt{\gamma}k\coth\sqrt{\gamma}k\right), \quad \alpha,\gamma\in\mathbb{R}$$

Covers *all* and *only* integrable instances of the theory

Conjecture that *all* higher order conditions are satisfied

1. Construct the charges of H in perturbation theory

Upshot: Charge densities given by

$$q_f(x) = \frac{1}{L}\sum_{k_1,k_2} f(k_1) e^{i(k_1-k_2)x}\psi^\dagger_{k_1}\psi_{k_2} + \frac{1}{L^2}\sum_{\boldsymbol{k}} g_f^{(4)}(\boldsymbol{k}) e^{i(k_1+k_2-k_3-k_4)x}\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi_{k_3}\psi_{k_4}$$
$$+\frac{1}{L^3}\sum_{\boldsymbol{k}} g_f^{(6)}(\boldsymbol{k}) e^{i(k_1+k_2+k_3-k_4-k_5-k_6)x}\psi^\dagger_{k_1}\psi^\dagger_{k_2}\psi^\dagger_{k_3}\psi_{k_4}\psi_{k_5}\psi_{k_6} + \dots$$

- local at order β

- local at higher order **only for integrable potentials**

- currents are found via

$$\partial_x J_f(x) = -i[H, q_f(x)] = -\partial_t q_f(x)$$

Strategy:

✔ 1. Construct the charges of H in perturbation theory

2. Build the “operatorial density of quasiparticles” using the them

2. Build the "operatorial density of quasiparticles" using the them

Define

$$n(k,x) := \frac{L}{2\pi}\frac{\partial}{\partial f(k)} q_f(x)$$

$$\langle n(k,x)\rangle = \rho(k)$$

Fulfils

$$\frac{2\pi}{L}\sum_k f(k) n(k,x) = q_f(x)$$

EV on a stationary state

$\Rightarrow$

$L \to \infty$

$\Updownarrow$

$$\int \mathrm{d}k\, f(k)\langle n(k,x)\rangle = \langle q_f(x)\rangle = \text{“}\int \mathrm{d}k\, f(k)\rho(k)\text{”}$$

2. Build the “operatorial density of quasiparticles” using the them

operatorial root density!

Define

$$n(k,x) := \frac{L}{2\pi}\frac{\partial}{\partial f(k)} q_f(x)$$

$$\langle n(k,x)\rangle = \rho(k)$$

Fulfils

$$\frac{2\pi}{L}\sum_k f(k) n(k,x) = q_f(x)$$

EV on a stationary state

$$\underset{L\to\infty}{\Rightarrow}$$

$$\Updownarrow$$

$$\int \mathrm{d}k\, f(k)\langle n(k,x)\rangle = \langle q_f(x)\rangle \;\text{“}\!=\int \mathrm{d}k\, f(k)\rho(k)\text{”}$$

2. Build the “operatorial density of quasiparticles” using the them

operatorial root density!

Define

$$n(k,x) := \frac{L}{2\pi}\frac{\partial}{\partial f(k)} q_f(x)$$

$$\langle n(k,x)\rangle = \rho(k)$$

Fulfils

$$\frac{2\pi}{L}\sum_k f(k) n(k,x) = q_f(x) \quad \overset{\text{EV on a stationary state}}{\underset{L\to\infty}{\Longrightarrow}} \quad \int \mathrm{d}k\, f(k)\langle n(k,x)\rangle = \langle q_f(x)\rangle = \text{“}\int \mathrm{d}k\, f(k)\rho(k)\text{”}$$

$$\Updownarrow$$

Define

$$J_n(k,x) := \frac{L}{2\pi}\frac{\partial}{\partial f(k)} J_f(x)$$

$$\partial_t n(k,x) + \partial_x J_n(k,x) = 0 \quad \overset{\text{EV on a stationary state}}{\Longrightarrow} \quad \text{“}\partial_t \rho(k,x,t) + \partial_x(v(k,x,t)\rho(k,x,t))\text{”} = 0$$

Operatorial GHD equation

2. Build the "operatorial density of quasiparticles" using the them

Explicitly

$$n(k,x) = \frac{1}{2\pi}\sum_q e^{i(k-q)x}\psi_k^\dagger \psi_q + \frac{1}{2\pi L}\sum_{\boldsymbol{k}} \frac{\partial}{\partial f(k)} g_f^{(4)}(\boldsymbol{k}) e^{i(k_1+k_2-k_3-k_4)x}\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}\psi_{k_4} + \dots$$

2. Build the "operatorial density of quasiparticles" using the them

Explicitly

$$n(k,x) = \frac{1}{2\pi}\sum_q e^{i(k-q)x}\psi_k^\dagger \psi_q + \frac{1}{2\pi L}\sum_{\boldsymbol{k}} \frac{\partial}{\partial f(k)} g_f^{(4)}(\boldsymbol{k}) e^{i(k_1+k_2-k_3-k_4)x}\psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}\psi_{k_4} + \dots$$

Remarks

1. For integrable cases one can explicitly show in PT

$$\langle n(k,x)\rangle\,|_m = \rho(k) + O(\beta^{m+1})$$

$$\langle j_n(k,x)\rangle\,|_m = v(k)\rho(k) + O(\beta^{m+1})$$

known from integrability

$$v(k) = k + \int \mathrm{d}q\, K(k-q)(v(q) - v(k)\rho(q)$$

2. Build the "operatorial density of quasiparticles" using the them

Explicitly

$$n(k,x) = \frac{1}{2\pi}\sum_q e^{i(k-q)x}\psi_k^\dagger\psi_q + \frac{1}{2\pi L}\sum_{\boldsymbol{k}} \frac{\partial}{\partial f(k)} g_f^{(4)}(\boldsymbol{k}) e^{i(k_1+k_2-k_3-k_4)x}\psi_{k_1}^\dagger\psi_{k_2}^\dagger\psi_{k_3}\psi_{k_4} + \dots$$

Remarks

1. For integrable cases one can explicitly show in PT

$$\langle n(k,x)\rangle|_m = \rho(k) + O(\beta^{m+1})$$

$$\langle j_n(k,x)\rangle|_m = v(k)\rho(k) + O(\beta^{m+1})$$

known from integrability

$$v(k) = k + \int \mathrm{d}q\, K(k-q)(v(q) - v(k)\rho(q)$$

2. The expectation value gives

$$\langle n(k,x)\rangle = \frac{1}{2\pi}\sum_q e^{i(k-q)x}\rho_2(k;q;t) + \frac{1}{2\pi L}\sum_{\boldsymbol{k}} \frac{\partial}{\partial f(k)} g_f^{(4)}(\boldsymbol{k}) e^{i(k_1+k_2-k_3-k_4)x}\rho_4(k_1,k_2;k_3,k_4;t) + \dots$$

$\Rightarrow \rho(k,x,t)$ obtained summing all the terms of the BBGKY hierarchy

2. Build the "operatorial density of quasiparticles" using the them

Explicitly

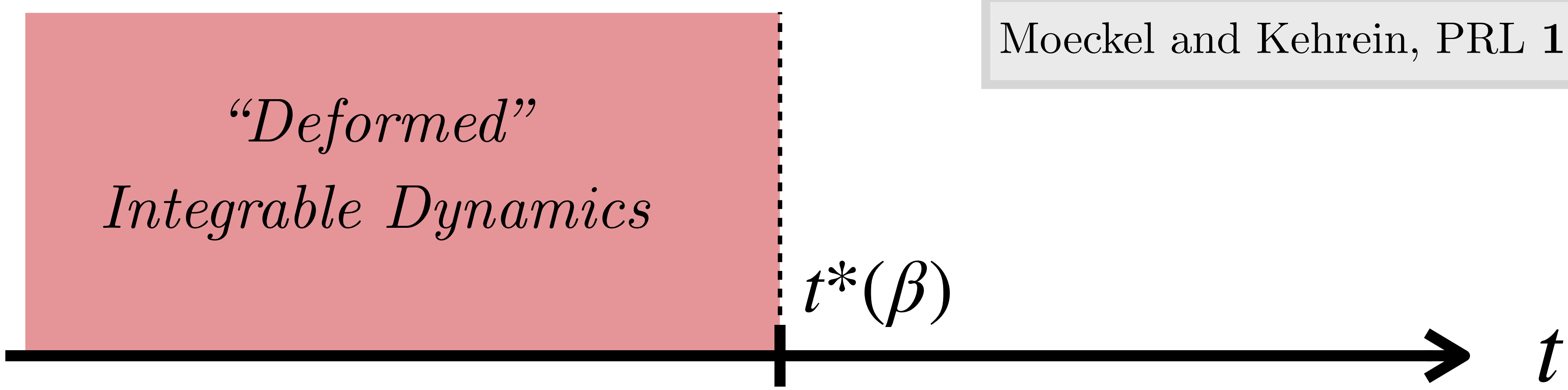
$$n(k,x) = \frac{1}{2\pi}\sum_q e^{i(k-q)x}\psi_k^\dagger\psi_q + \frac{1}{2\pi L}\sum_{\boldsymbol{k}} \frac{\partial}{\partial f(k)} g_f^{(4)}(\boldsymbol{k}) e^{i(k_1+k_2-k_3-k_4)x}\psi_{k_1}^\dagger\psi_{k_2}^\dagger\psi_{k_3}\psi_{k_4} + \dots$$

Remarks

3. $n(k,x)$ *always local* at first order in β

- The local dynamics is constrained on an intermediate window: *"Prethermalization"*

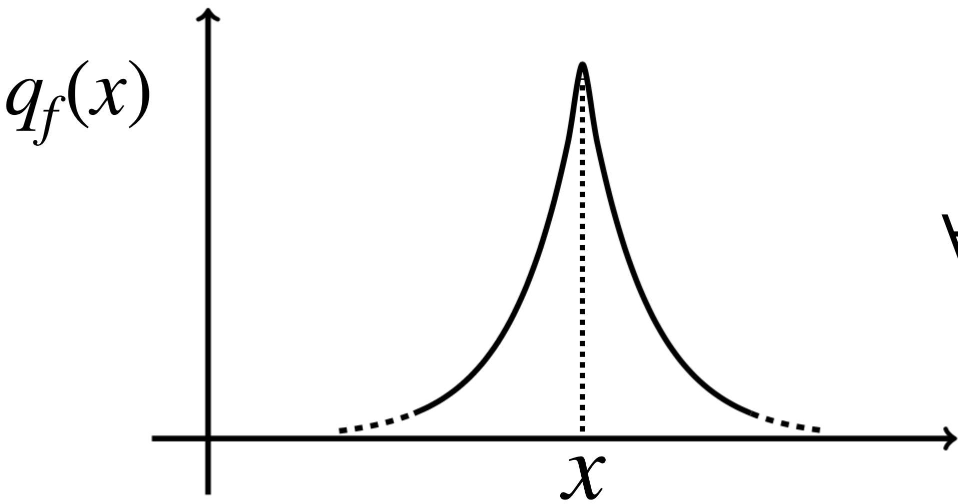
Moeckel and Kehrein, PRL **100**, 175702 (2008).



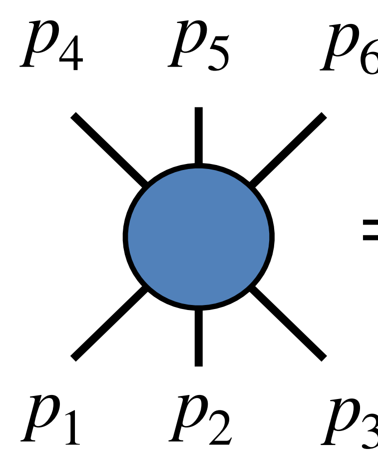
- The transition can be understood by using Fermi Golden rule $t^*(\beta) \sim \beta^{-2}$

BB et al. PRL **115**, 180601 (2015); Durnin et al., PRL **127**, 130601 (2021); Lopez-Piqueres et al. PRB **103**, L060302 (2021); ...

Summary



$\forall f \quad \Leftrightarrow$



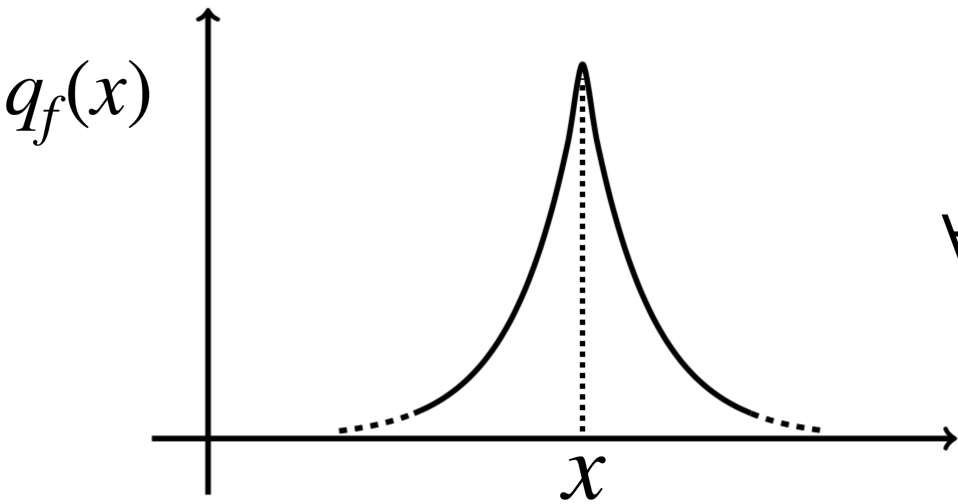
$= 0 \qquad \{p_1, p_2, p_3\} \neq \{p_4, p_5, p_6\}$

Perturbative construction of *conserved charges* and *currents* in *weakly interacting fermions*

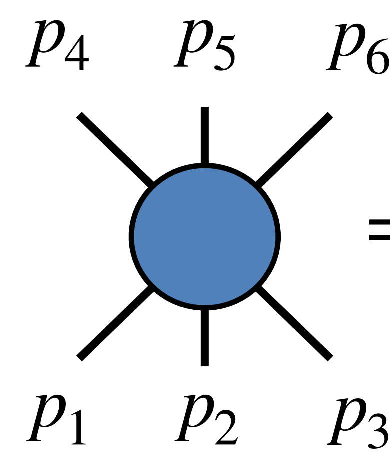
1. Only for integrable potentials *all* charges are local at second order

2. "Operatorial root density" written as a sum of monomials in the fermionic operators

Outlook



$\forall f \quad \Leftrightarrow$



$= 0 \qquad \{p_1, p_2, p_3\} \neq \{p_4, p_5, p_6\}$

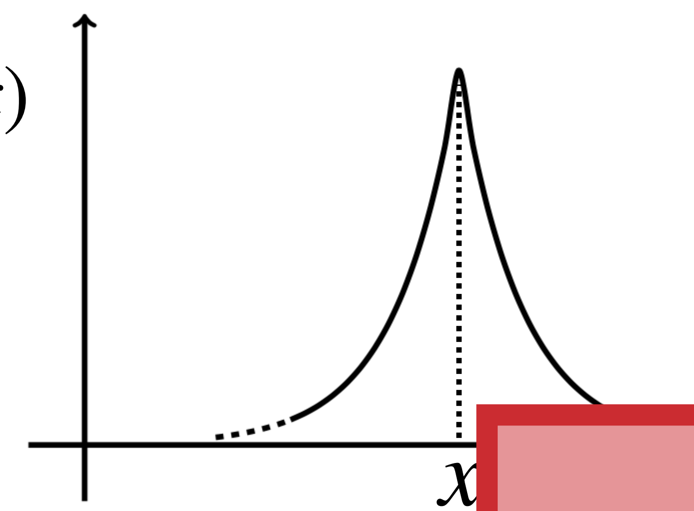
Perturbative construction of *conserved charges* and *currents* in *weakly interacting fermions*

1. Extend the treatment to systems with bound states

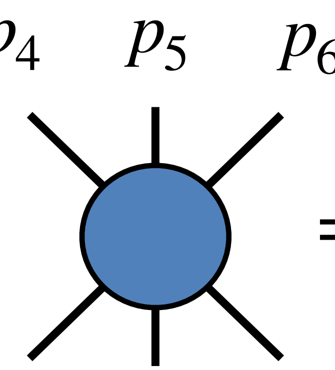
2. Systematic treatment of finite-time corrections to GHD equations

3. Systematic treatment of integrability breaking

Outlook



$\forall f \quad \Leftrightarrow$



$= 0 \qquad \{p_1, p_2, p_3\} \neq \{p_4, p_5, p_6\}$

Perturbative construc ... *y interacting fermions*

Thank you!

1. Extend the treatment to systems with bound states

2. Systematic treatment of finite-time corrections to GHD equations

3. Systematic treatment of integrability breaking