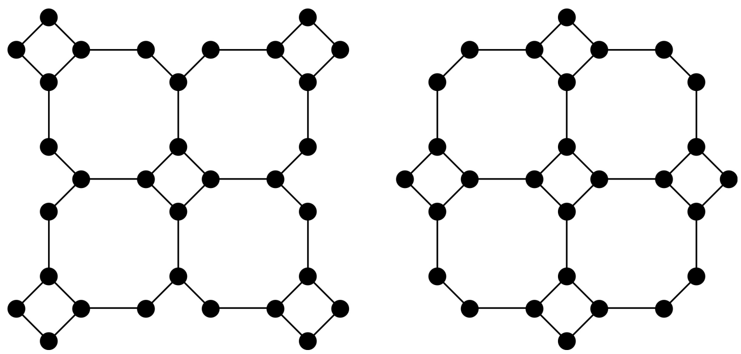
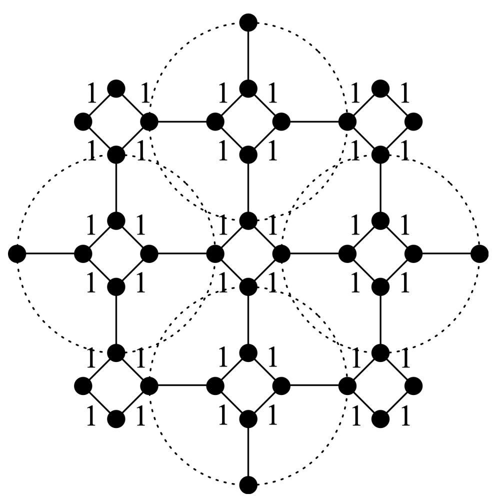
Solutions to Problems for Lecture #1

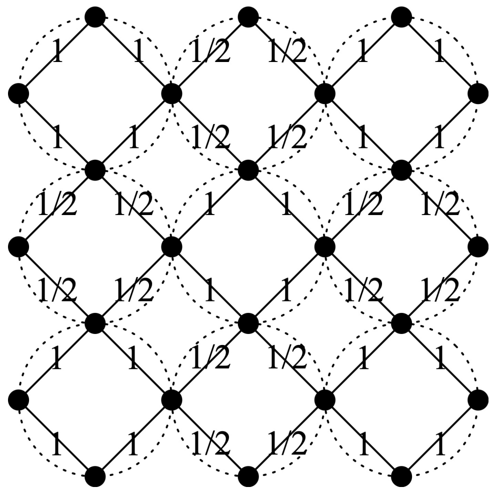
1. *Add appropriate edges of weight 0 and weight 1 and use Δ-factors to count the (perfect) matchings of the two graphs shown below.*



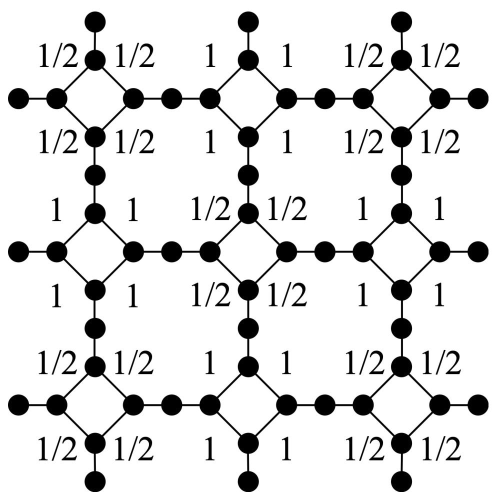
As indicated in the hint, we begin by adding some vertices and edges as shown:



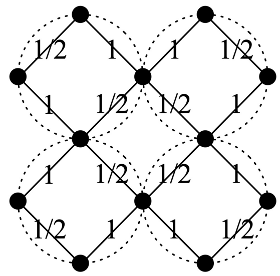
and applying four reverse spider moves:



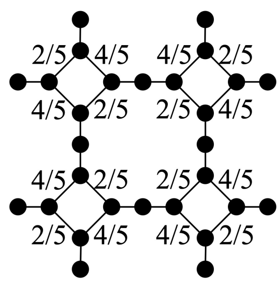
This gives us a weighted Aztec diamond graph of order 3. Now we do $a, b, c, d \to A, B, C, D$ transformations inside the nine circles:



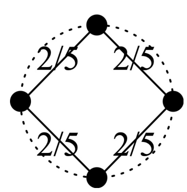
After we peel off edges and contract, we get a weighted Aztec diamond graph of order 2.



Now we do $a, b, c, d \to A, B, C, D$ transformations inside the four circles:

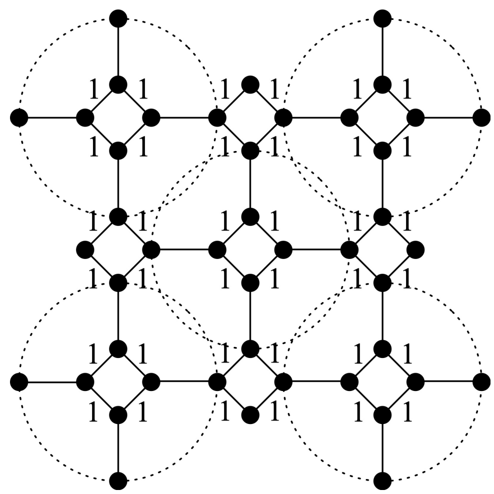


After we peel off edges and contract, we get a weighted Aztec diamond graph of order 1.



The Δ-factors we encountered along the way were 2, 2, 2, 2 (for the reverse spider moves); 2, 1/2, 2, 1/2, 2, 1/2, 2, 1/2, 2 (in the order 3 Aztec diamond graph); 5/4, 5/4, 5/4, 5/4 (in the order 2 Aztec diamond graph); and 8/25 (in the order 1 Aztec diamond graph). Multiplying all these factors together we get 25.

For the second problem, we follow the same procedure.


1 1 1 1 1 1
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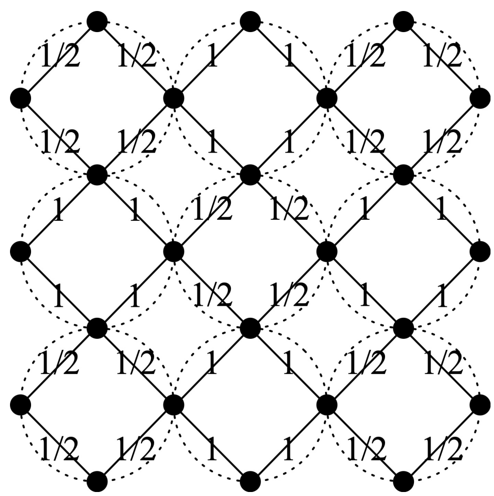
1/2 1/2 1 1 1/2 1/2
1/2 1/2 1 1 1/2 1/2
1 1 1/2 1/2 1 1
1 1 1/2 1/2 1 1
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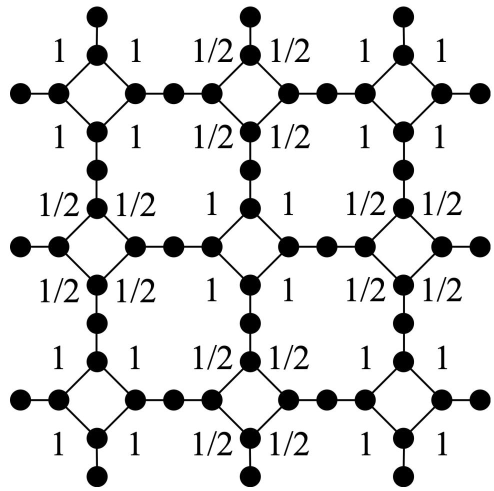
1/2 1/2 1 1 1/2 1/2


1 1 1/2 1/2 1 1
1 1 1/2 1/2 1 1
1/2 1/2 1 1 1/2 1/2
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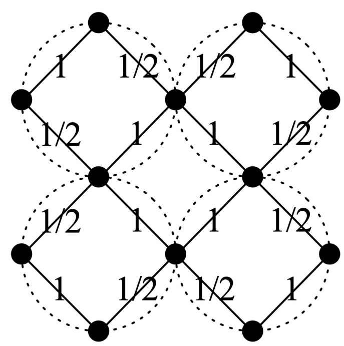
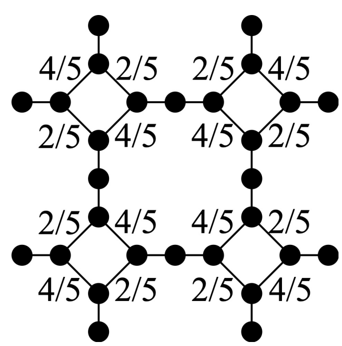
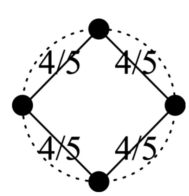
1 1 1/2 1/2 1 1


1 1/2 1/2 1
1/2 1 1 1/2
1/2 1 1 1/2
1 1/2 1/2 1

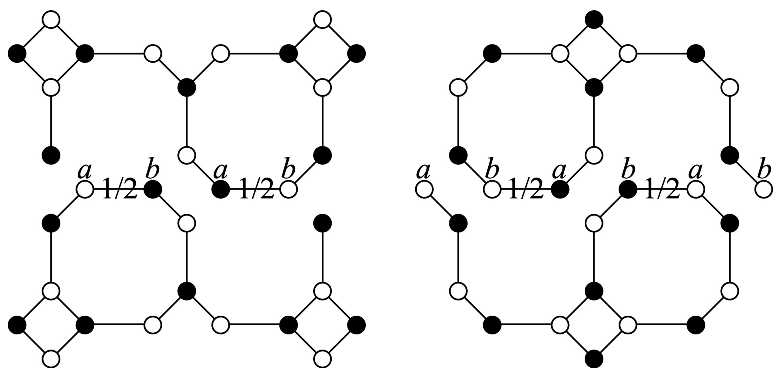

4/5 2/5 2/5 4/5
2/5 4/5 4/5 2/5
2/5 4/5 4/5 2/5
4/5 2/5 2/5 4/5


4/5 4/5
4/5 4/5

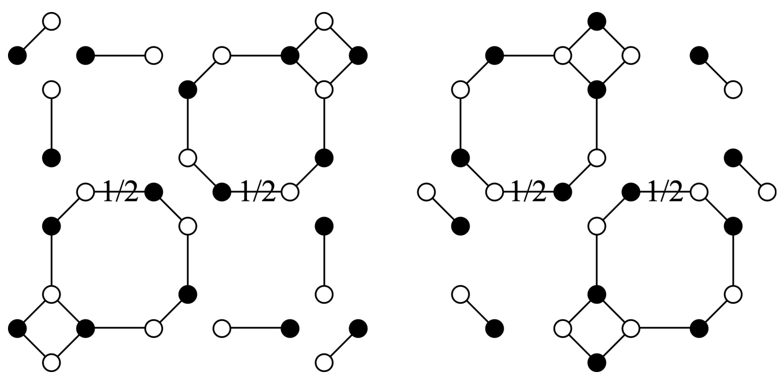
The Δ-factors are 2, 2, 2, 2, 2 for the reverse spider moves; $1/2$, 1, $1/2$, 1, $1/2$, 1, $1/2$, 1, $1/2$ for the next step; $5/4$, $5/4$, $5/4$, $5/4$ for the next step; and lastly $32/25$. The product is 50.

2. *Use Ciucu factorization to count the matchings of the graphs from problem 1.*

Cutting the graphs as dictated by the definition of G^+ and G^- we get



There are many forced edges:



We find that when the forced edges are removed, we are left with subgraphs consisting of an octagon that shares an edge with a square. This graph has three matchings of respective weights 1, 1, and $1/2$, which sum to $5/2$. So Ciucu's formula gives us $2^2\,(5/2)(5/2) = 25$ and $2^3\,(5/2)(5/2) = 50$ as the respective numbers of matchings of the two graphs.