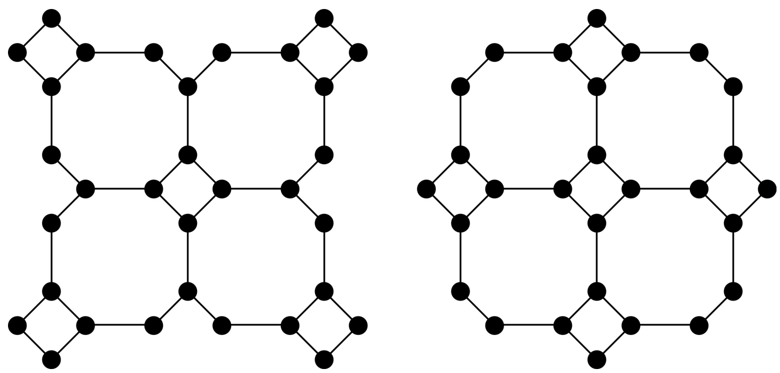
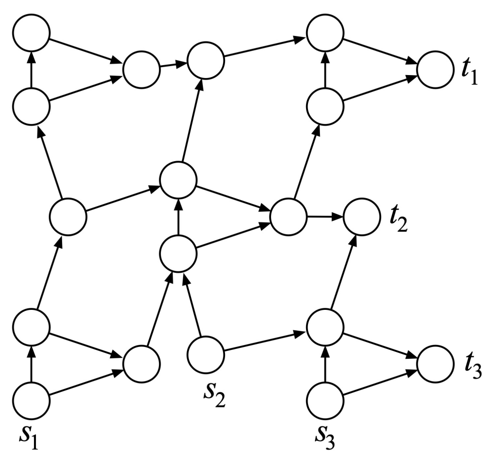
Solutions to Problems for Lecture #2

Note: Lecture notes are posted at http://jamespropp.org/its2.pdf.

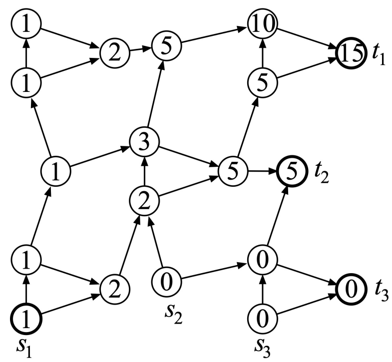
3. *Use Lindström's lemma to count the (perfect) matchings of the two fortress graphs we've looked at.*

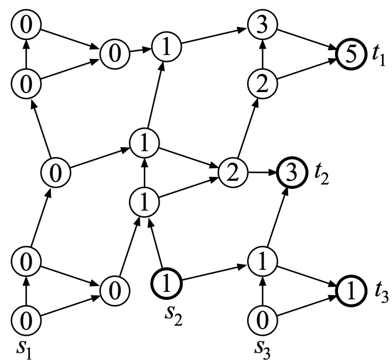


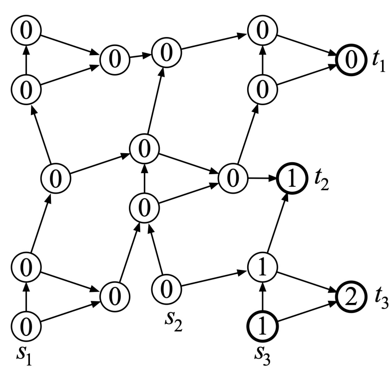
I showed during lecture that we can reduce the problem of counting matchings of the first fortress graph to the problem of counting nonintersecting triples of paths in the directed graph



We count paths from s_1 to all t_j, then from s_2 to all t_j, and then from s_3 to all t_j:





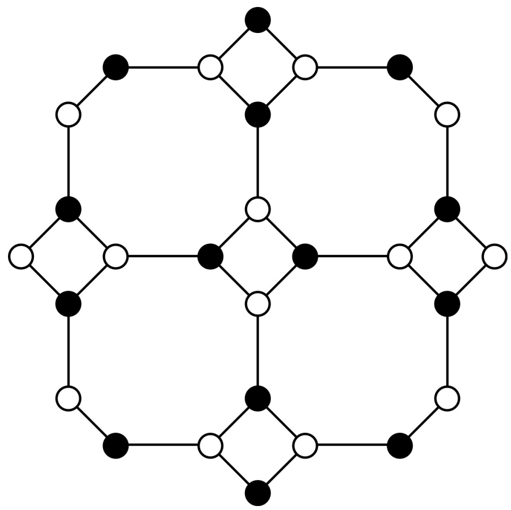


The number of nonintersecting triples of paths is therefore

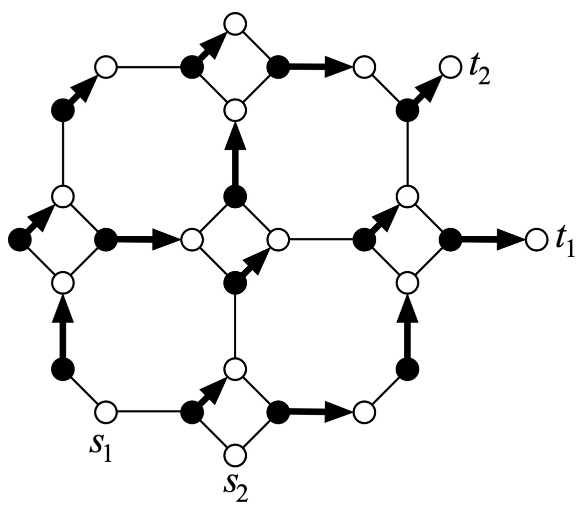
$$\begin{vmatrix} 15 & 5 & 0 \\ 5 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 25$$

which is therefore the number of matchings of the first fortress graph.

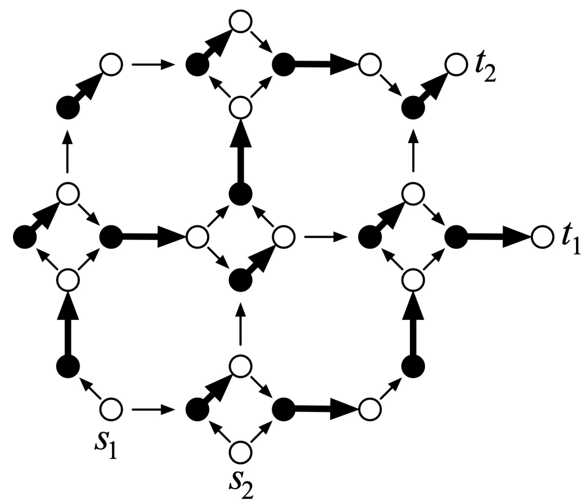
For the second fortress graph, there are a number of solutions; here's one that gets away with using a 2-by-2 determinant rather than a 3-by-3 determinant. 2-color the vertices as shown:



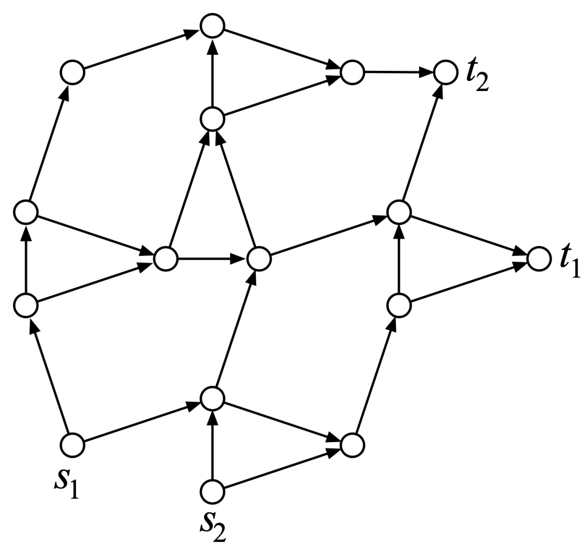
Impose a black-to-white near-matching with two white source vertices and two extra white vertices serving as target vertices:



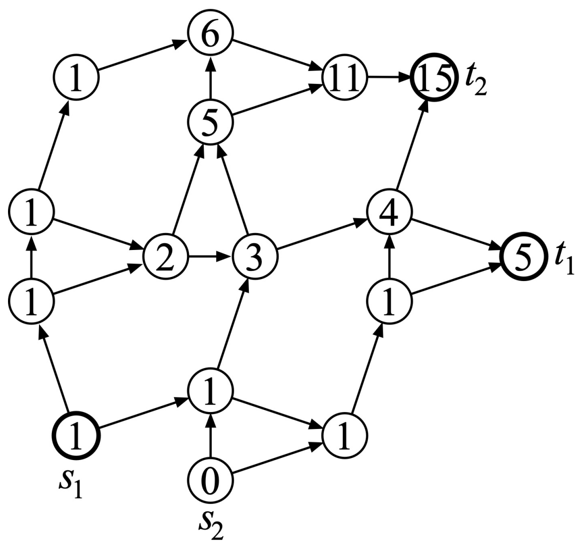
Add arrows corresponding to all the edges that can occur in a matching (besides the ones represented as arrows in the previous image), directed from white to black:

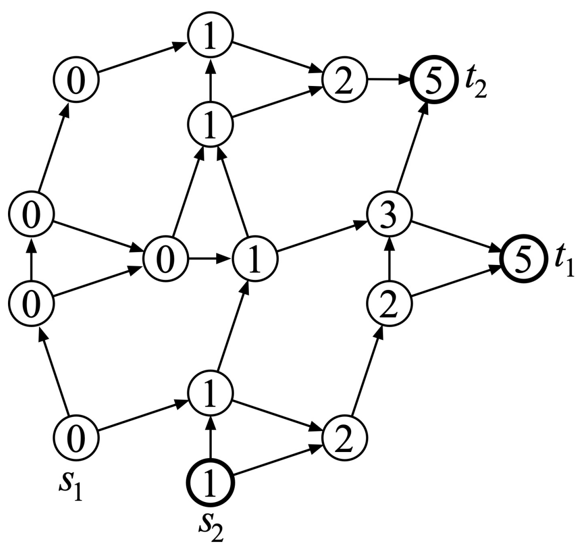


Shrink away the black vertices so that we get a directed graph on just the white vertices, two of which are sources s_1, s_2 and two of which are targets t_1, t_2:



We count paths from s_1 to t_1 and t_2, then from s_2 to t_1 and t_2:





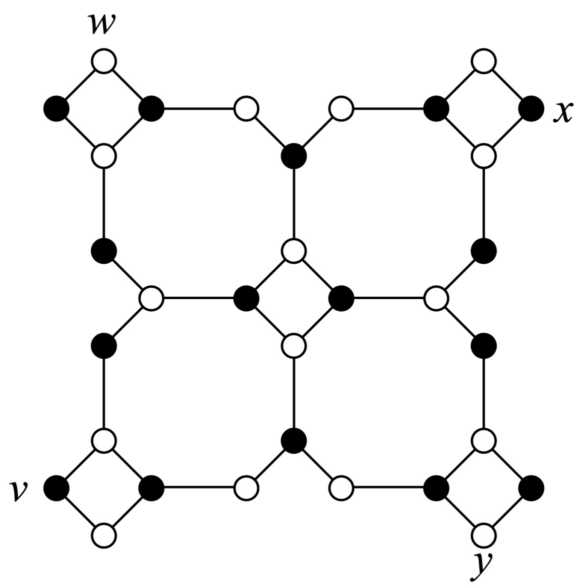
The number of nonintersecting triples of paths is therefore

$$\begin{vmatrix} 15 & 5 \\ 5 & 5 \end{vmatrix} = 50$$

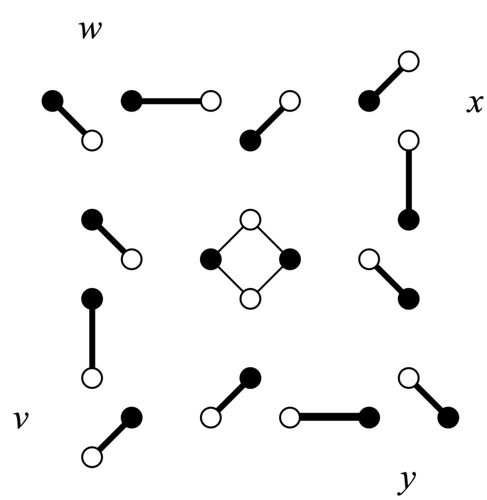
which is therefore the number of matchings of the second fortress graph.

4. *Use Kuo condensation to count the matchings of those two graphs.*

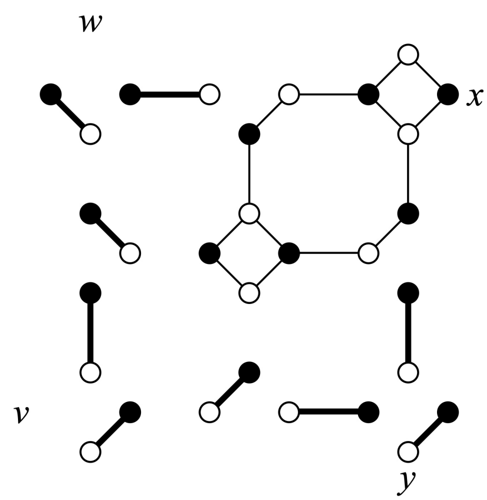
For the first graph (call it G_1), here are good vertices to use as v, w, x, y:



When we remove the four vertices, many edges are forced; when those forced edges are removed, what remains is a 4-cycle that can be matched in 2 ways.

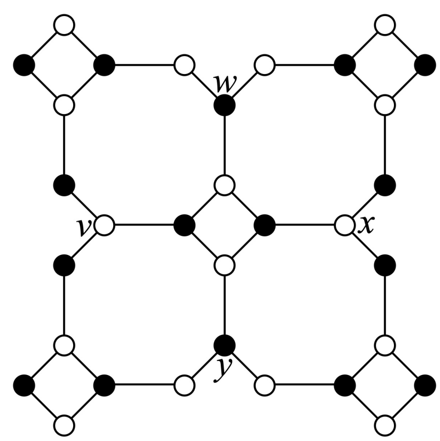


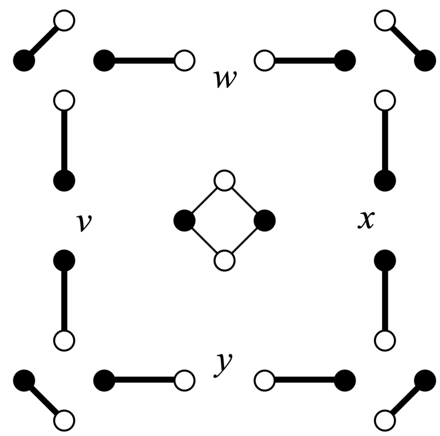
When we remove just v and w, we get this graph, which is easily seen to have 5 matchings:

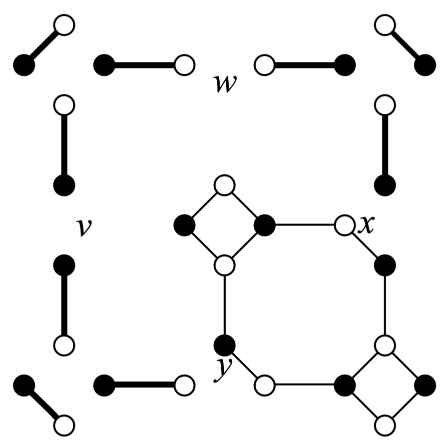


By symmetry, we also get graphs with 5 matchings when we remove x and y, v and y, or w and x. Kuo's relation tells us that $M(G_1)$ (the number of matchings of this graph) satisfies $(M(G_1))(2) = (5)(5) + (5)(5) = 50$, so $M(G_1) = 25$.

Here's another solution:

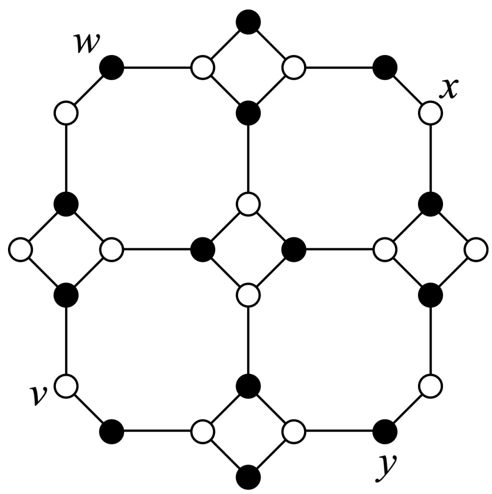




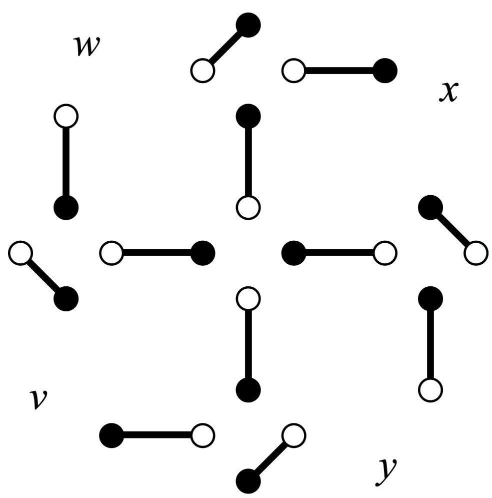


As in the previous solution, removing v, w, x, and y gives a graph with just 2 matchings, while removing v and w (or w and x, or x and y, or y and w) gives a graph with 5 matchings, so we get $M(G_1) = (5^2 + 5^2)/2 = 25$.

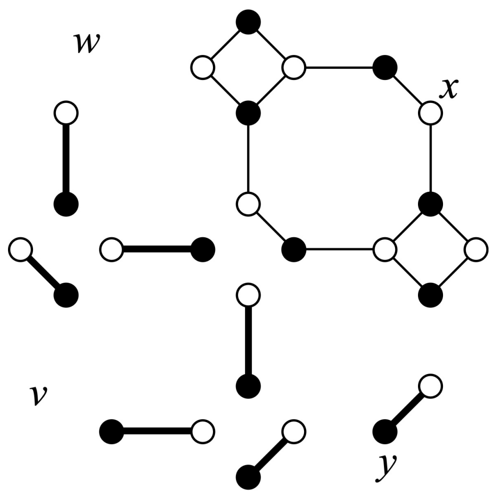
For the second graph (call it G_2), here are good vertices to use as v, w, x, y:



When we remove the four vertices and all forced edges, nothing is left; the graph can be matched in only 1 way.



When we remove just v and w, we get a graph with 5 matchings:



By symmetry, we also get graphs with 5 matchings when we remove x and y, v and y, or w and x. Kuo's relation tells us that $M(G_2)$ (the number of matchings of this graph) satisfies $(M(G_2))(1) = (5)(5) + (5)(5) = 50$, so $M(G_2) = 50$.