Combinatorics and Exact Enumeration in Dimer Models

Lecture #2

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ITS Summer School on Dimers
August 16, 2023

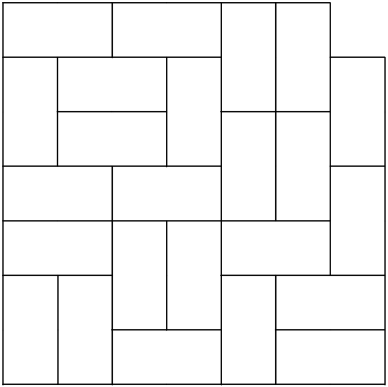
Slides for this talk and the group work assignment are at

http://faculty.uml.edu/jpropp/its2.pdf and

http://faculty.uml.edu/jpropp/its-P2.pdf

I. Temperley's bijection

There is a bijection between domino tilings of the $(2m-1)$-by-$(2n-1)$ rectangle with a corner removed and spanning trees of the m-by-n grid graph.



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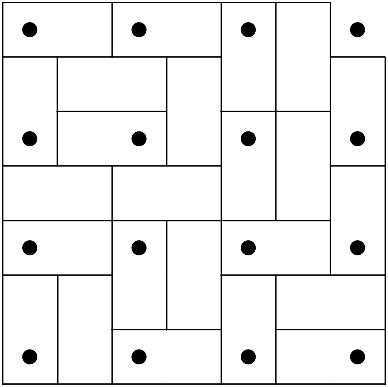

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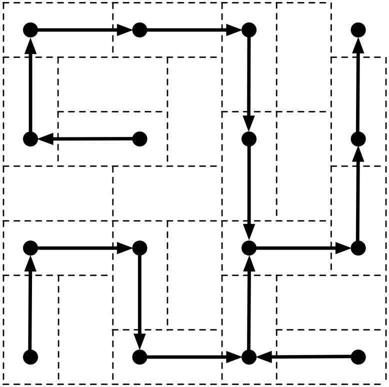

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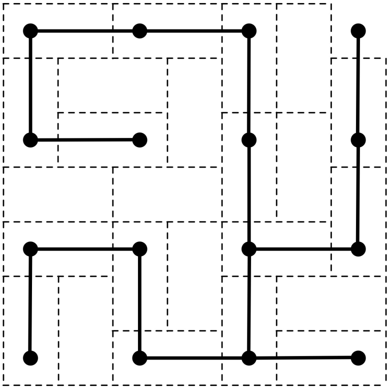


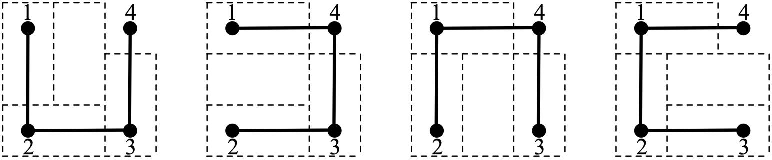
Counting spanning trees

Matrix-tree theorem: Given a simple graph G with vertices $v_1, \ldots, v_n$, let A_{ij} $(1 \leq i, j \leq n)$ be -1 if v_i and v_j are adjacent and let A_{ij} be $\deg i$ if $i = j$ (and 0 otherwise).

Then for all $1 \leq i \leq n$, the $(n-1)$-by-$(n-1)$ matrix obtained from A by crossing out the ith row and ith column of A has determinant equal to the number of spanning trees of G.

Counting spanning trees

For instance, to count the domino tilings of a 3-by-3 square with a corner removed, use Temperley's bijection to replace them by counting spanning trees of a 2-by-2 grid graph, and then evaluate a 3-by-3 subdeterminant of the associated 4-by-4 matrix.

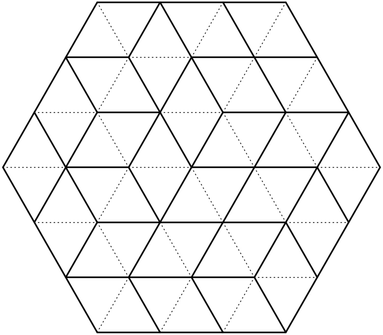


$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 4$$

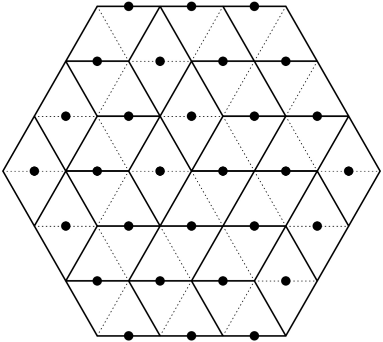
Dimer covers and spanning trees

For generalizations of Temperley's bijection, see Kenyon et al. (2000) and Kenyon-Sheffield (2004).

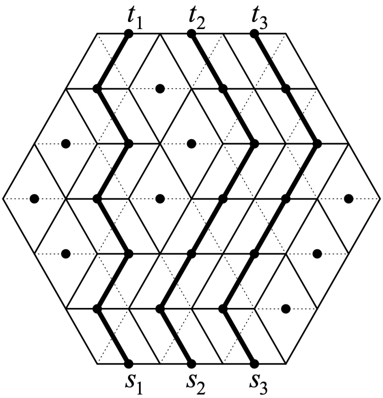
II. Lindström's Lemma



Lindström's Lemma



Lindström's Lemma



Lindström's Lemma

Given an acyclic finite directed graph G with n source nodes $s_1, \ldots, s_n$ and n terminal nodes $t_1, \ldots, t_n$, let $A_{i,j}$ $(1 \leq i, j \leq n)$ be the number of paths from s_i to t_j.

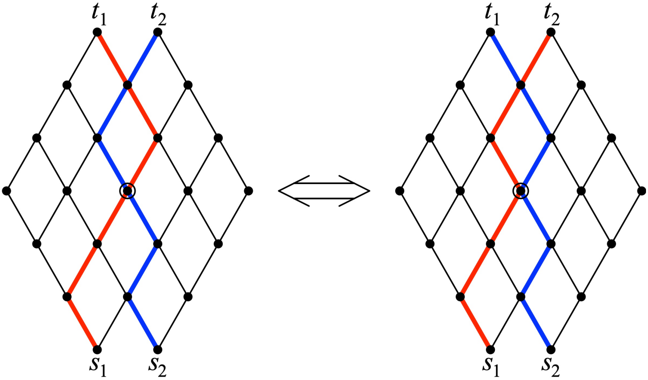
Suppose that the only way to have n nonintersecting paths joining the initial nodes to the terminal nodes is to join s_i to t_i for $1 \leq i \leq n$.

Then the determinant of A (aka the Gessel-Viennot matrix) is the number of families of n pairwise-nonintersecting lattice paths from the source nodes to the terminal nodes.

Lindström's Lemma

Special case $n = 2$: the number of pairs of nonintersecting paths joining s_1 to t_1 and s_2 to t_2 equals $A_{1,1}A_{2,2} - A_{1,2}A_{2,1}$.

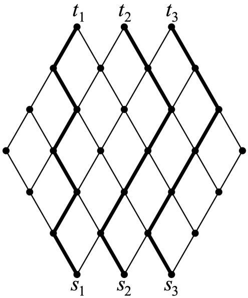
Proof-idea: Use path-switching to show that the number of pairs of **intersecting** paths joining s_1 to t_1 and s_2 to t_2 equals $A_{1,2}A_{2,1}$.



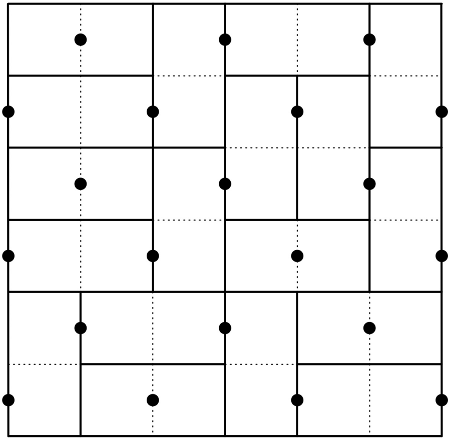
Lindström's Lemma

Application of Lindström's Lemma: The number of lozenge tilings of the hexagon of side 3 is

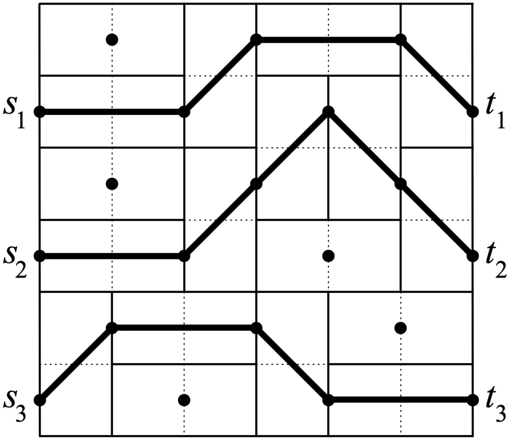
$$\begin{vmatrix} \binom{6}{3} & \binom{6}{2} & \binom{6}{1} \\ \binom{6}{4} & \binom{6}{3} & \binom{6}{2} \\ \binom{6}{5} & \binom{6}{4} & \binom{6}{3} \end{vmatrix} = 980$$



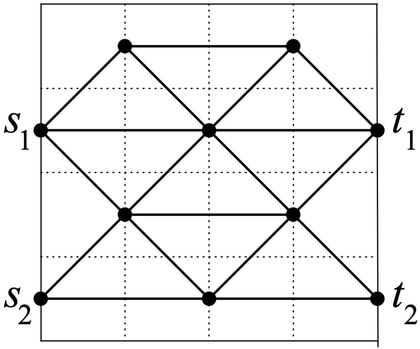
Domino tilings and Randall paths



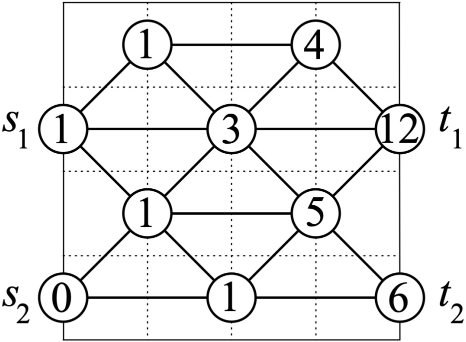
Domino tilings and Randall paths



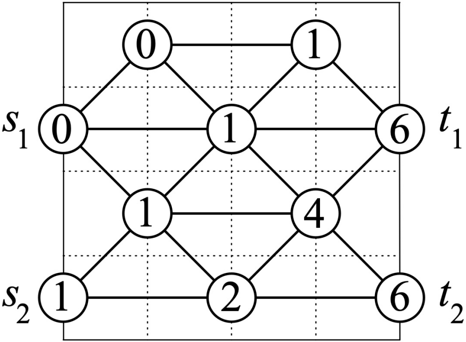
Domino tilings and Randall paths



Domino tilings and Randall paths



Domino tilings and Randall paths



Domino tilings and Randall paths

Hence the number of domino tilings of the 4-by-4 square is

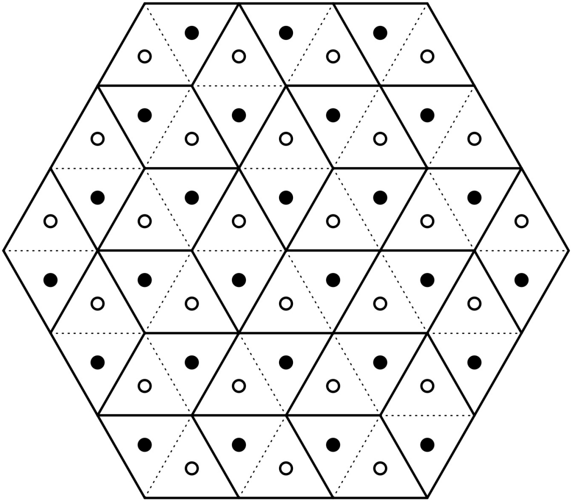
$$\begin{vmatrix} 12 & 6 \\ 6 & 6 \end{vmatrix} = 36$$

The double-dimer perspective

A way to unify these two constructions is to imagine we are superimposing two dimer configurations, one fixed (though not quite a dimer cover) and one varying. The fixed near-cover will omit some vertices and include some extra vertices outside the graph under consideration.

The fixed near-matching should be "extremal" in the sense that if you take its symmetric difference with any (perfect) matching of the graph, you'll get a union of paths and doubled edges (no cycles of length > 2).

The double-dimer perspective



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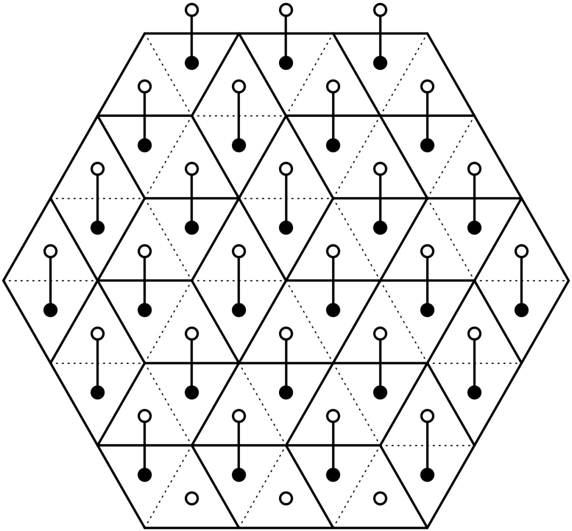

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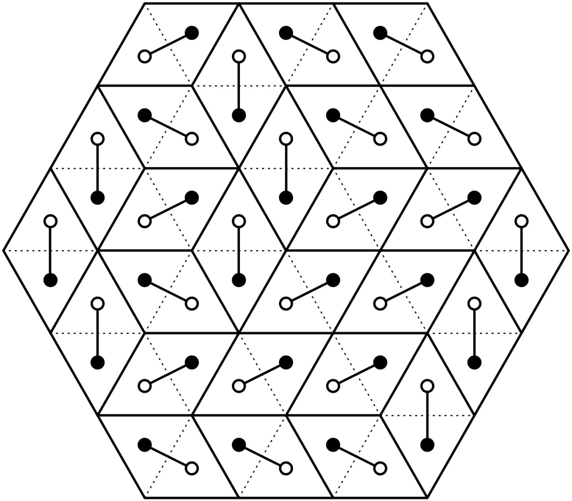

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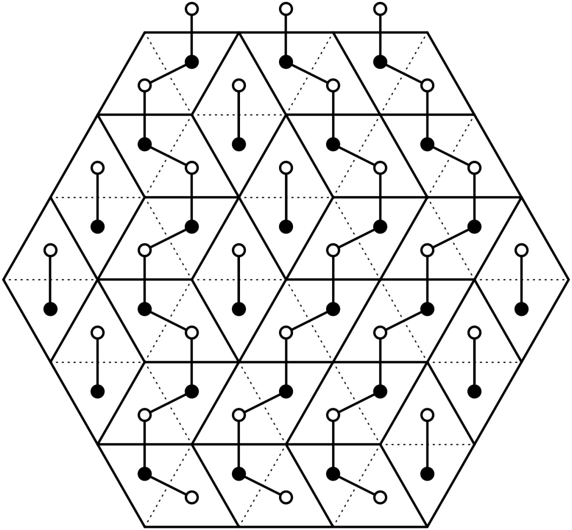

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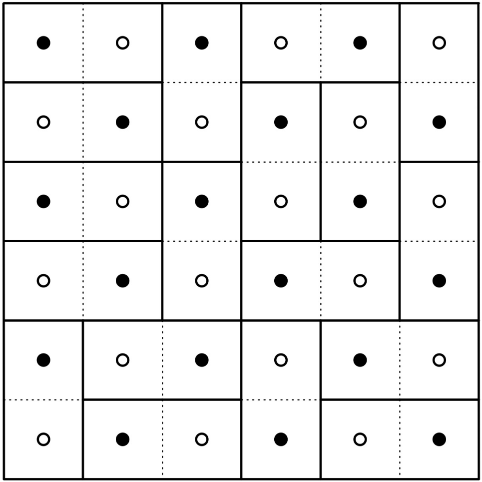

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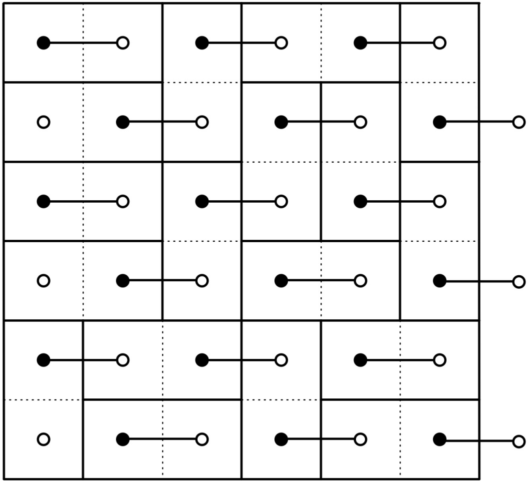

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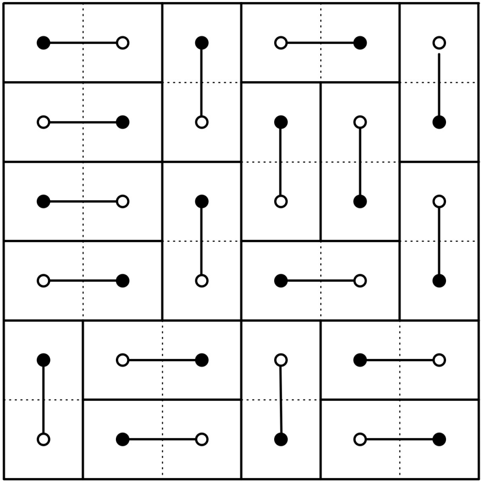

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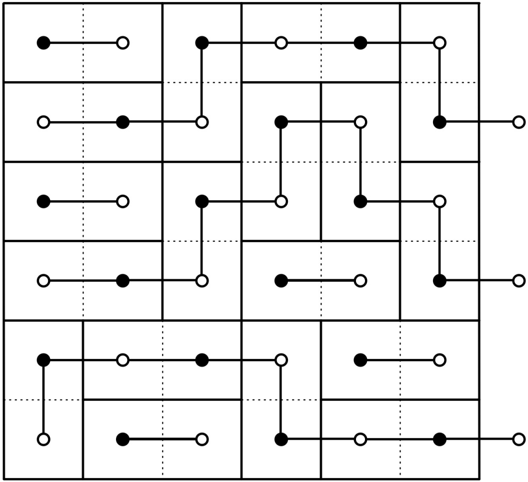
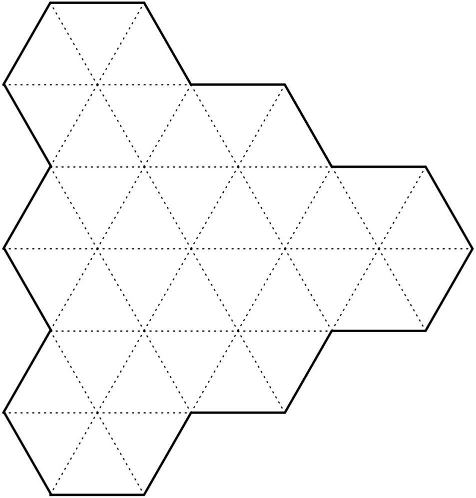


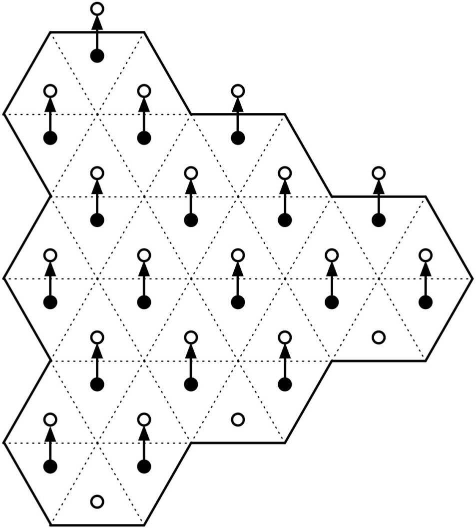
The directed double-dimer perspective

Apply Lindström's Lemma to lozenge tilings of this region:



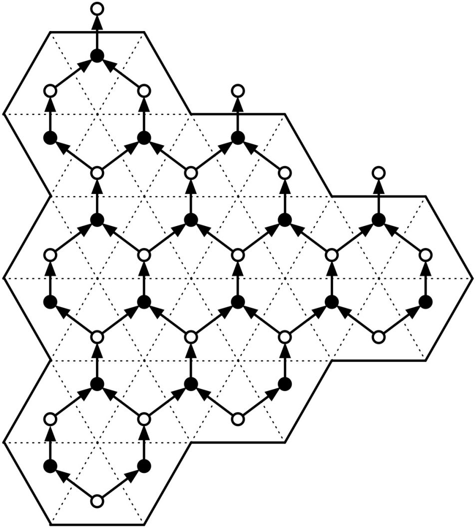
The directed double-dimer perspective

Orient a fixed extremal near-cover from black to white:



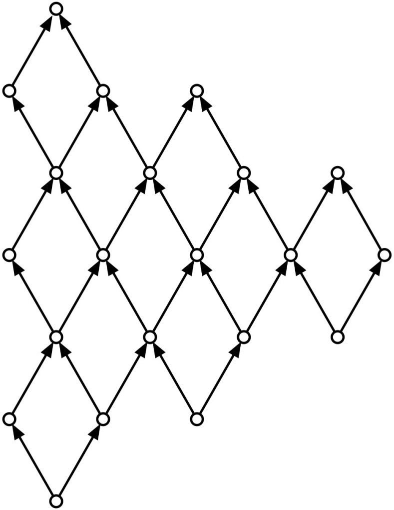
The directed double-dimer perspective

Orient the varying cover from white to black:

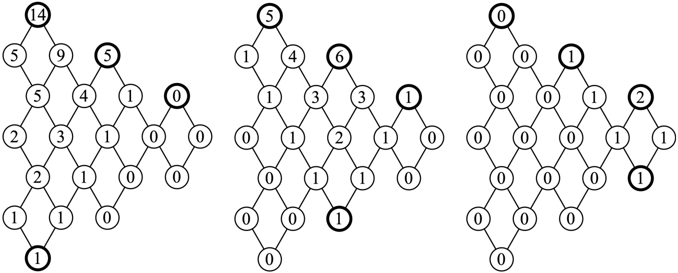


The directed double-dimer perspective

Shrink away the black vertices:



The directed double-dimer perspective

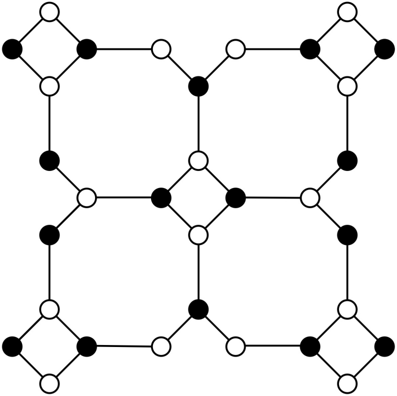


The number of tilings is

$$\begin{vmatrix} 14 & 5 & 0 \\ 5 & 6 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 104$$

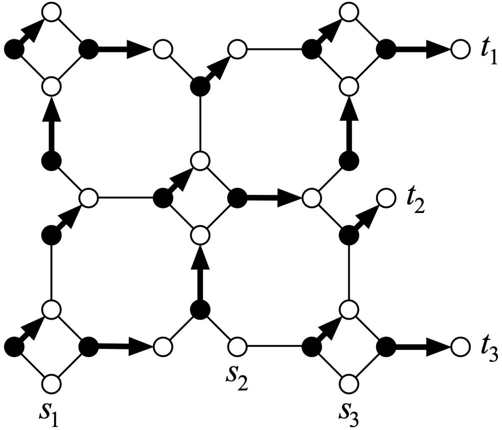
Exercise 3

Use Lindström's lemma to count matchings of the two fortress graphs we've looked at. Here's one of them:



Exercise 3

Orient a fixed extremal near-cover from black to white:



Exercise 3

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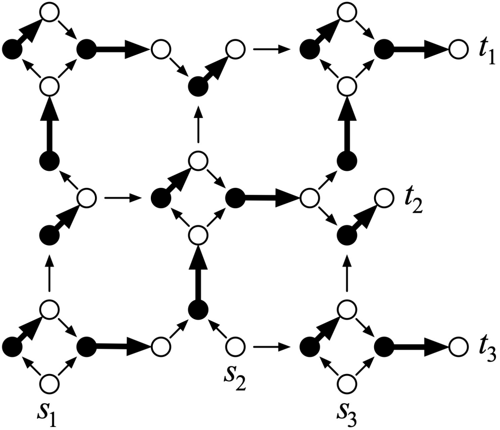
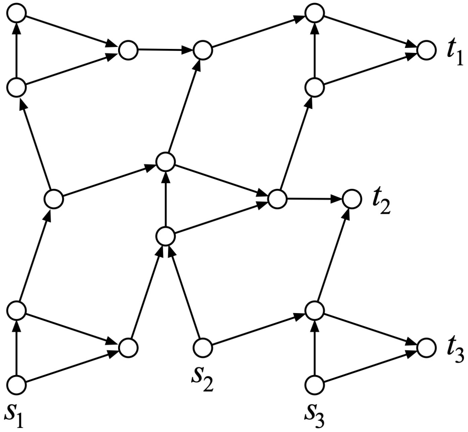


Exercise 3

Shrink away the black vertices:



III. Permanents and determinants

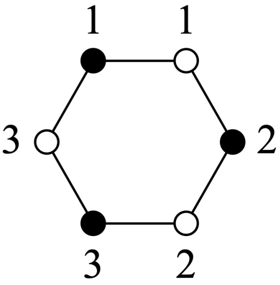
The number of perfect matchings of a bipartite planar graph G with $2n$ vertices (n black, n white) is equal to the permanent of the n-by-n bipartite adjacency matrix A whose i, jth entry is 1 or 0 according to whether the ith black vertex is adjacent to the jth white vertex.

This seems useless, since permanents (unlike determinants) are hard to compute.

But Kasteleyn showed if G is planar, then one can tamper with the signs of the nonzero elements of A (or, more generally, replace them by complex numbers of norm 1) in such a way that in the resulting matrix K, all nonzero contributions to the determinant interfere constructively, so that $|\det K|$ is the number of matchings.

Determinants and matchings

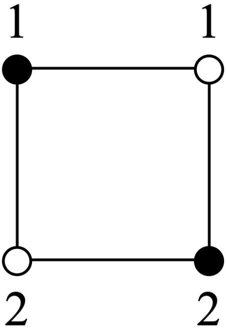
Sometimes you can just take $K = A$ (e.g., if G is a 6-cycle, or more generally a $4k+2$-cycle):



$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2$$

Determinants and matchings

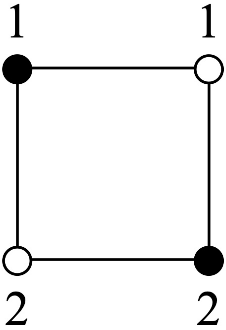
But usually you need to make changes (e.g., if G is a 4-cycle, or more generally a $4k$-cycle):



$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Determinants and matchings

But usually you need to make changes (e.g., if G is a 4-cycle, or more generally a $4k$-cycle):



$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

Determinants and matchings

Kasteleyn proved that if G is a bipartite plane graph, there is always a way to assign signs to its edges so that the product of the signs around a face is $+1$ or -1 according to whether the number of edges around the face is 2 (mod 4) or 0 (mod 4).

This gives a matrix whose determinant has magnitude equal to the number of perfect matchings of G.

(Kasteleyn used $2n$-by-$2n$ matrices; Percus noticed that n-by-n matrices could be used instead.)

Determinants and matchings

You can also use complex numbers as weights, as long (reading around each face-cycle of edges) the product of the weights of the 1st, 3rd, 5th, ... edges is either equal to, or the negative of, the product of the weights of the 2nd, 4th, 6th, ... edges, according to whether the number of edges around the face is 2 (mod 4) or 0 (mod 4).

For instance, in a square grid graph, we could put weight 1 on all horizontal edges and weight i on all vertical edges.

Determinants and matchings

Kasteleyn showed that for m, n even, the number of domino tilings of an m-by-n board is

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right),$$

the product of the eigenvalues of K.

For my version of his proof, see Propp 2014.

The inverse Kasteleyn matrix

The entries of K^{-1} are important in statistical mechanics. Specifically, if the ith black vertex is adjacent to the jth white vertex, then (the magnitude of) the i,j entry of K^{-1} equals the probability that a random matching of the graph matches the ith black vertex to the jth white vertex.

This is an easy consequence of the cofactor formula for the inverse of a matrix.

The rate at which these correlations decay is hugely important for understanding random dimer covers.

Non-planar graphs

If G can be embedded on a surface of genus g, then we can write the number of matchings of G as a sum of 4^g determinants.

This is especially helpful when we look at dimers on a torus ($g = 1$); when the torus is large, the associated dimer model (aka the dimer model with periodic boundary conditions) is a good stand-in for the physicist's limit of "infinite size".

Non-bipartite graphs

If G is a planar graph that isn't bipartite, then we need to use Pfaffians instead of determinants; see Kuperberg 1998 and its references.

Relationship to Lindström matrices

The last two methods of counting matchings (Lindström and Kasteleyn) are closely related; indeed, one can "continuously" interpolate between them.

See Kuperberg 1998.

IV. Kuo condensation

Let $G = (V, E)$ be a plane bipartite graph with vertices colored black and white, let F be a face of G (possibly the infinite face), with vertices v, w, x, and y appearing in cyclic order around F, with v and x black and w and y white.

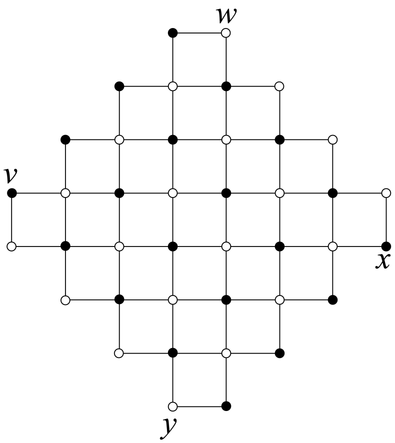
For any set $V' \subseteq V$, let $G - V'$ denote the induced subgraph of G on the vertex set $V - V'$, and let $M_{V'}$ denote the number of matchings of this subgraph, so that for instance M_ϕ is the number of matchings of G.

Theorem (Kuo 2004):

$$M_\phi M_{\{v,w,x,y\}} = M_{\{v,w\}} M_{\{x,y\}} + M_{\{v,y\}} M_{\{w,x\}}$$

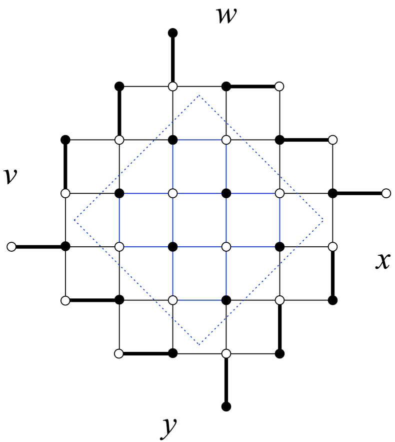
Aztec diamonds

Let $\mathrm{AD}(n)$ be the Aztec diamond graph of order n, and let G be an $\mathrm{AD}(4)$, with vertices v, w, x, y (black, white, black, white) as shown.



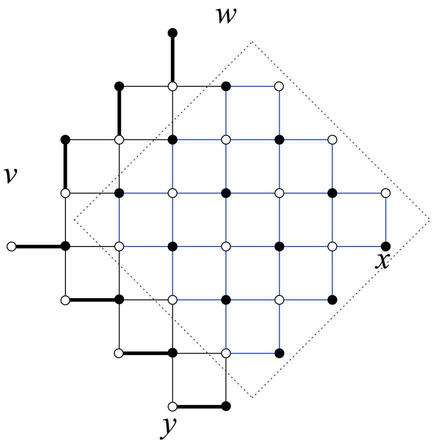
Aztec diamonds

When v, w, x, and y are all removed, the edges around the border are all either forced to be included or forced to be excluded, leaving an $\mathrm{AD}(2)$ in the middle.



Aztec diamonds

When v and w (or x and y, or v and y, or w and x) are removed, and forcibly included/excluded edges are removed, what's left is an $\mathrm{AD}(3)$.



Aztec diamonds

Let A_n denote the number of matchings of $\mathrm{AD}(n)$.

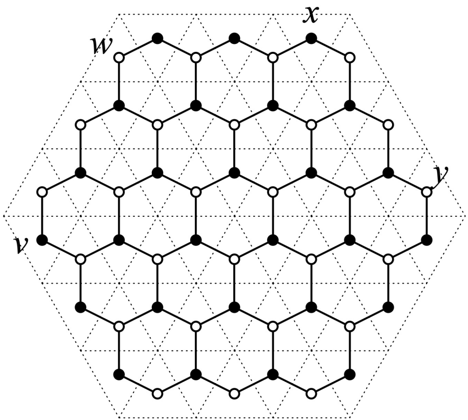
Kuo's formula, applied to $G = \mathrm{AD}(4)$, yields

$$A_4 A_2 = A_3 A_3 + A_3 A_3$$

More generally, we have $A_{n+1} A_{n-1} = 2A_n^2$ for all n, which (combined with initial conditions $A_0 = 1$ and $A_1 = 2$) yields $A_n = 2^{n(n+1)/2}$.

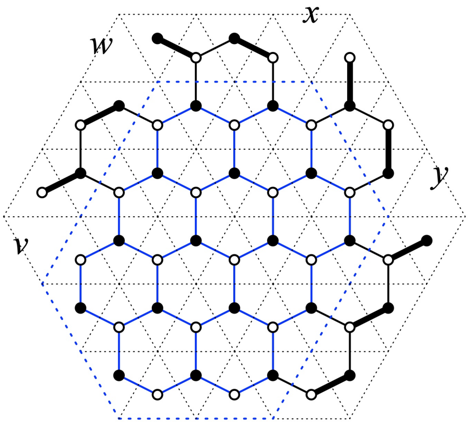
Hexagons

Let $\mathrm{Hex}(a, b, c)$ be the dual of the hexagon with sides a, b, c, a, b, c, and let G be a $\mathrm{Hex}(3, 3, 3)$, with vertices v, w, x, y (black, white, black, white) as shown.



Hexagons

When v, w, x, and y are all removed, the edges around the border are all either forced to be included or forced to be excluded, forming a $\mathrm{Hex}(3, 2, 2)$.



Hexagons

When v and w are removed, what's left is a $\mathrm{Hex}(3,3,2)$.
When x and y are removed, what's left is a $\mathrm{Hex}(3,2,3)$.
When v and y are removed, what's left is a $\mathrm{Hex}(4,2,2)$.
When w and x are removed, what's left is a $\mathrm{Hex}(2,3,3)$.
So Kuo's condensation formula tells us that

$$H_{3,3,3}H_{3,2,2} = H_{3,3,2}H_{3,2,3} + H_{4,2,2}H_{2,3,3}$$

where $H_{a,b,c}$ denote the number of matchings of $\mathrm{Hex}(a,b,c)$.

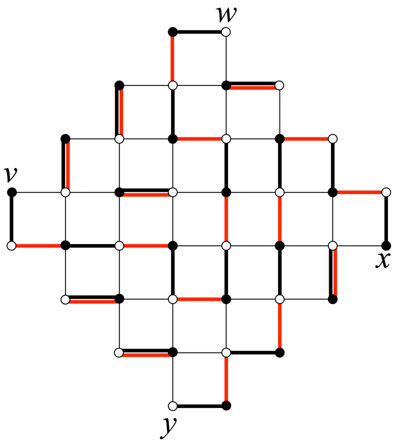
More generally,

$$H_{a,b,c}H_{a,b-1,c-1} = H_{a,b,c-1}H_{a,b-1,c} + H_{a+1,b-1,c-1}H_{a-1,b,c}\ ;$$

this identity (combined with suitable initial conditions) yields Macdonald's formula for $H_{a,b,c}$ by induction on $a+b+c$.

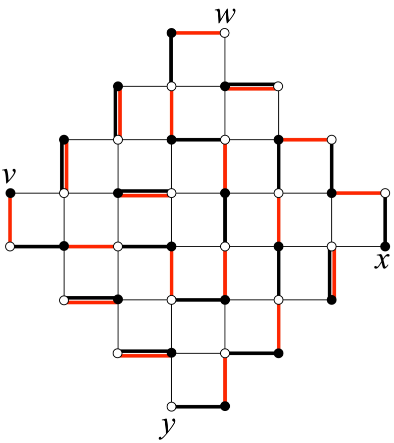
Idea behind proof of Kuo's formula

Suppose we have a matching of G (black) superimposed with a matching of $G - \{v, w, x, y\}$ (red). Suppose there are black-red-$\cdots$-black paths joining v to w and x to y.



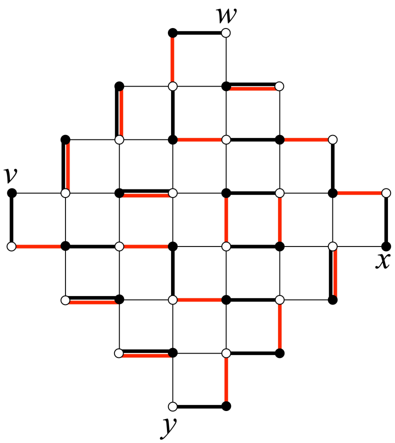
Idea behind proof of Kuo's formula

Swapping red with black along the path joining v to w, we get a matching of $G - \{v, w\}$ (black) superimposed with a matching of $G - \{x, y\}$ (red).



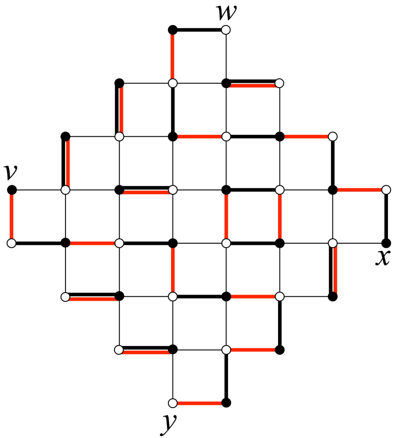
Idea behind proof of Kuo's formula

On the other hand, suppose there are black-red-$\cdots$-black paths joining v to y and w to x. (Note that there can't be paths joining v to x and w to y for topological reasons.)



Idea behind proof of Kuo's formula

Swapping red with black along the path joining v to y, we get a matching of $G - \{v, y\}$ (black) superimposed with a matching of $G - \{w, x\}$ (red).



Variations

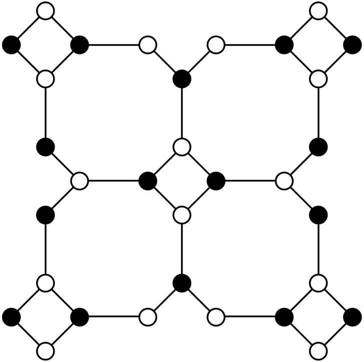
The four vertices could be black, black, white, white or even 3 of one color and 1 of the other color.

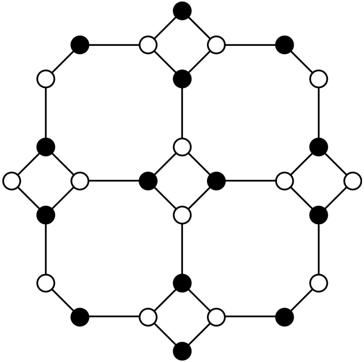
The graph need not be bipartite.

See Yan et al. 2005.

Exercise 4

Use Kuo condensation to count matchings of the two fortress graphs.





References with links

R. Kenyon, J. Propp, and D. Wilson, Trees and matchings, Electronic Journal of Combinatorics **7** R25 (2000)

R. Kenyon and S. Sheffield, Dimers, tilings, and trees, Journal of Combinatorial Theory, Series B **92**, 295–317 (2004)

E. Kuo, Applications of graphical condensation for enumerating matchings and tilings, Theoretical Computer Science **319**, 29–57 (2004)

G. Kuperberg, An exploration of the permanent-determinant method (1998)

J. Propp, Dimers and dominoes (2014)

W. Yan, Y. Yeh, and F. Zhang, Graphical condensation of plane graphs: A combinatorial approach, Theoretical Computer Science **349**, 452–461 (2005)

. . . and these slides and homework 2.