Combinatorics and Exact Enumeration in Dimer Models

Lecture #3

James Propp, UMass Lowell

ITS Summer School on Dimers
August 18, 2023

All slides are now available

Notes:

http://faculty.uml.edu/jpropp/its1.pdf

http://faculty.uml.edu/jpropp/its2.pdf

http://faculty.uml.edu/jpropp/its3.pdf

Problems and solutions for assignment #1:

http://faculty.uml.edu/jpropp/its-P1.pdf

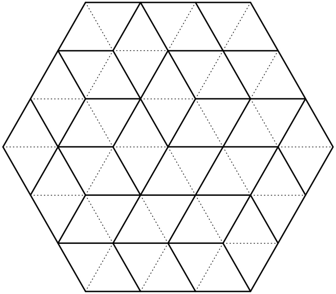
http://faculty.uml.edu/jpropp/its-S1.pdf

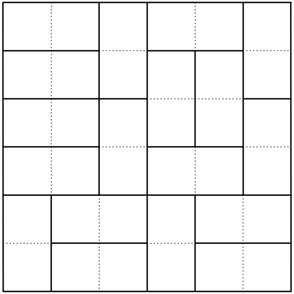
Problems and solutions for assignment #2:

http://faculty.uml.edu/jpropp/its-P2.pdf

http://faculty.uml.edu/jpropp/its-S2.pdf

Height functions





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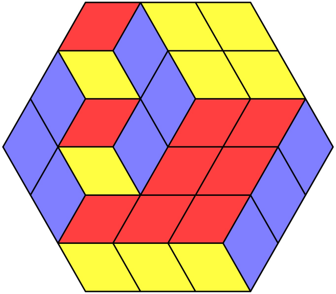
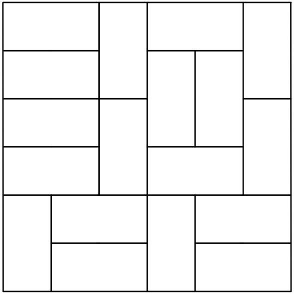
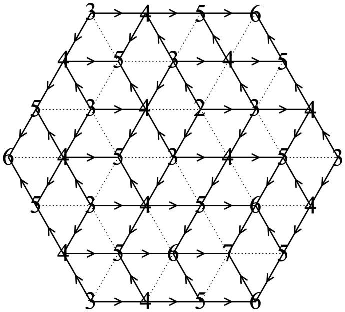
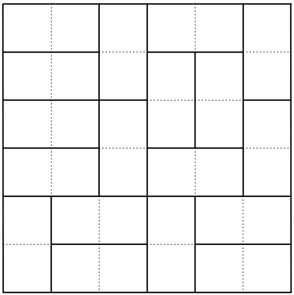


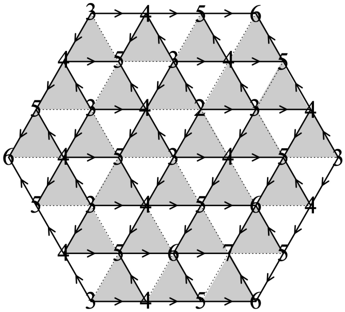


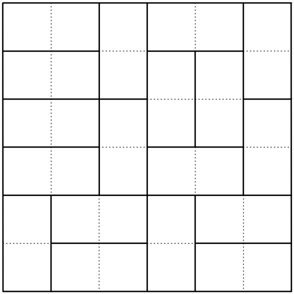
Height functions



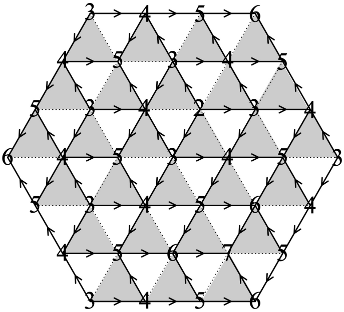


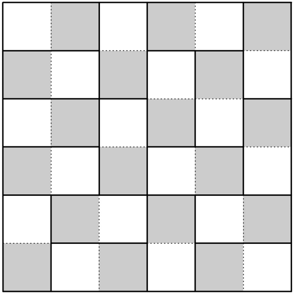
Height functions



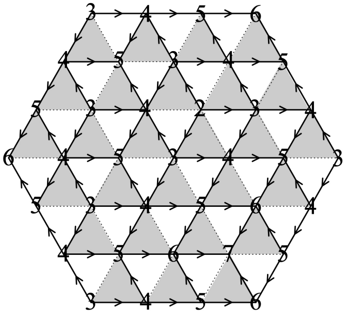


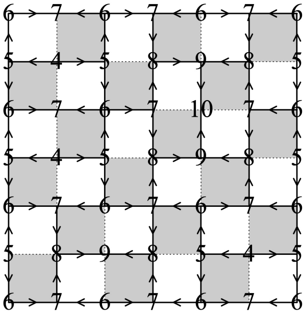
Height functions





Height functions





Height functions

Suppose G is a bipartite plane graph, each of whose edges belongs to at least one matching of G (but doesn't belong to all matchings of G).

Then we can represent each matching M by a real-valued function $H = H_M$ on the faces of G, called a height function, which is uniquely defined (modulo the value it takes on the unbounded face) and from which the matching can be reconstructed.

There are actually many ways to do this, all essentially equivalent; I'll give a uniform definition for all graphs G, though for specific graphs G one might want to tweak it (e.g., scaling H up by a constant to make it integer-valued).

Height functions

Let μ_0 be a positive fractional matching of G; that is, a positive function on the edge-set of G with the property that for every vertex v, $\sum_e \mu(e) = 1$ where e is summed over all edges containing v.

(E.g., $\mu_0(e)$ could be the proportion of matchings of G that contain e.)

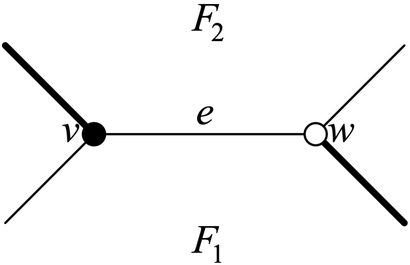
Let F_1 and F_2 be adjacent faces sharing an edge e; say the endpoints of e are the black vertex v and the white vertex w.

Then we take

$$H_M(F_2) - H_M(F_1) = \begin{cases} \mu_0(e) & \text{if } e \notin M, \\ \mu_0(e) - 1 & \text{if } e \in M \end{cases}$$

Height functions

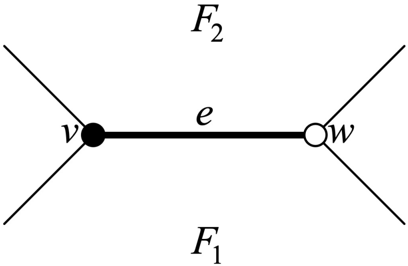
When the edge e is NOT in the matching M:



$$H_M(F_2) - H_M(F_1) = \mu_0(e)$$

Height functions

When the edge e IS in the matching M:



$$H_M(F_2) - H_M(F_1) = \mu_0(e) - 1$$

Height functions

Local consistency: If we hold the black vertex v fixed and take the neighbors of v in counterclockwise order $(w_1, w_2, \ldots, w_n)$ so that the edges $e_1, e_2, \ldots$ pivot counterclockwise around v, the heights of the successive faces $F_1, F_2, \ldots$ increase by $\mu_0(e)$ except when $e \in M$, in which case they increase by $\mu_0(e) - 1$ (i.e., they decrease by $1 - \mu_0(e)$). The total height increase as e makes one turn around v is

$$\left(\sum_e \mu_0(e) \right) - 1 = 1 - 1 = 0$$

where e is summed over all edges containing v.

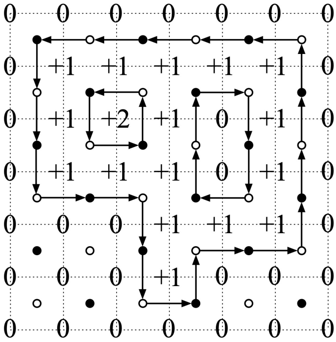
Height functions

Local consistency implies global consistency.

We also have uniqueness, since every face can be reached from the unbounded face by a succession of such pivots.

Height differences and flows

Orient M_1 from black to white and M_2 from white to black and remove all 2-cycles (doubled edges), obtaining a flow.

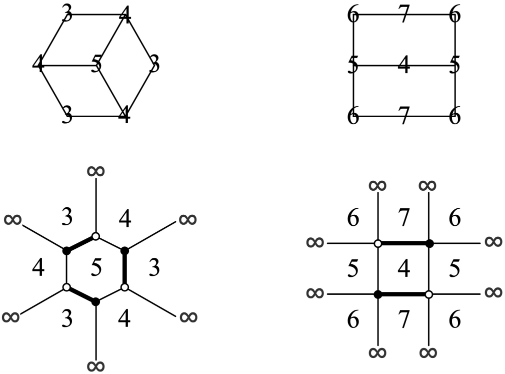


Then the difference between the height of face F under M_1 and the height of face F under M_2 is essentially the winding number of the flow around F.

Tilings and matchings

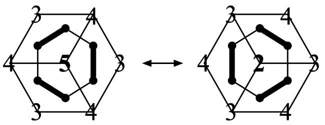
In tilings, the height function lives on the set of vertices.

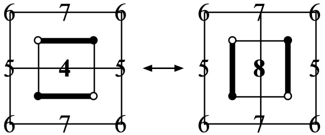
In matchings, the height function lives on the set of faces. Sometimes it's handy to have several unbounded faces, with different (fixed) heights.



Face twists

Using the height function, we can endow the set of matchings with the structure of a distributive lattice whose covering relations correspond to "face twists".





All matchings can be reached from one another by face twists. See James Propp 2002, Lattice structure for orientations of graphs (https://arxiv.org/abs/math/0209005).

Face twists and Kasteleyn

This twist-move picture sheds light on Kasteleyn's theorem, in particular, the role played by the product of the weights of the 1st, 3rd, 5th, ... edges around a face divided by the product of the weights of the 2nd, 4th, 6th, ... edges around the face (call this (*)).

The determinant of a matrix $A_{i,j}$ is a sum of terms of the form

$$\mathrm{sign}(\pi)\ A_{1,\pi(1)} A_{2,\pi(2)} \cdots A_{n,\pi(n)}$$

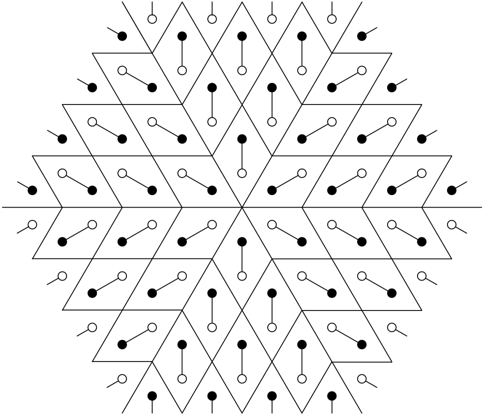
When you twist around a face with $2k$ sides, you compose π with a k-cycle, which multiplies the sign by $(-1)^{k+1}$.

You also multiply $A_{1,\pi(1)} \cdots A_{n,\pi(n)}$ by (*), so if Kasteleyn's condition is satisfied, the two multiplications cancel each other, and the terms of the determinant interfere constructively.

Matchings of the plane

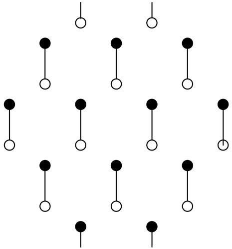
In finite graphs, all dimer states are related by face moves. In infinite graphs, things are very different.

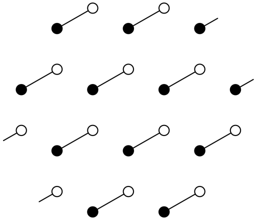
E.g., in the infinite hexagon graph, there are matchings that are rigid (no face moves are possible).

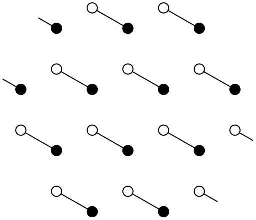


Matchings of the plane

There are three rigid matchings that are invariant under all the translation-symmetries of the graph.

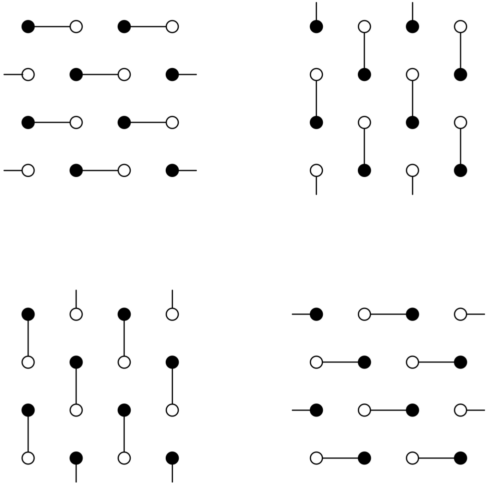






Matchings of the plane

Likewise, there are four rigid matchings of the square grid that are invariant under all color-preserving translation-symmetries of the graph.



Matchings of the plane

Such matchings, restricted to finite patches, are extremal in the sense required in Lecture #2.

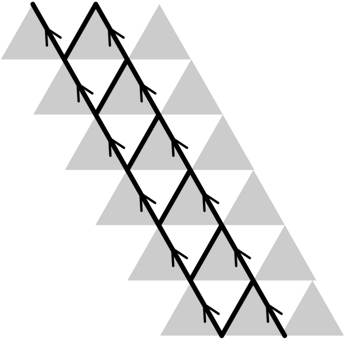
That is because there is a direction in which the height is increasing as rapidly as it can.

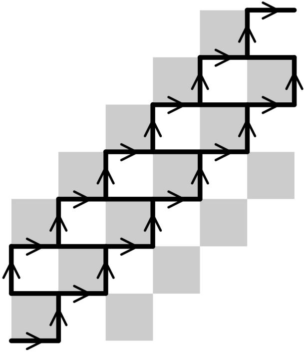
Every face lies on an infinite path of faces along which the height is monotone increasing.

Frozen regions in random tilings of large finite regions tend to exhibit local behavior given by such extremal matchings of the infinite graph.

Tilings in the plane

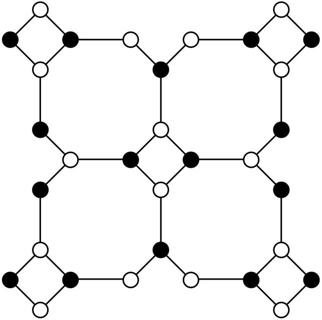
In the dual picture, an extremal tiling has the property that every vertex lies on an infinite path of vertices along which the height is monotone: the faces adjoining the path are black on one side and white on the other.





Matchings of the plane

Let's look again at fortress graphs, this time in the dual picture ("diabolo tilings").



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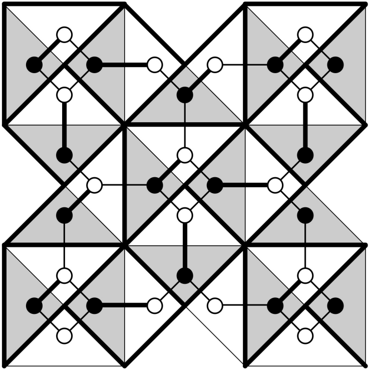

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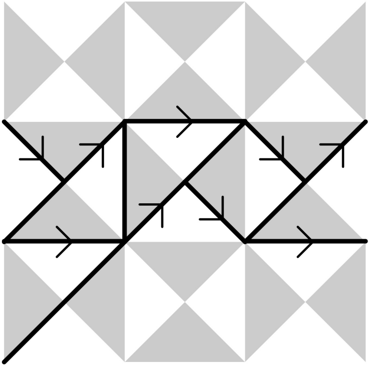
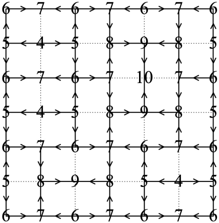
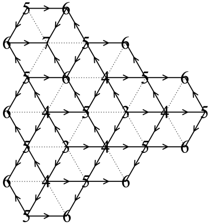


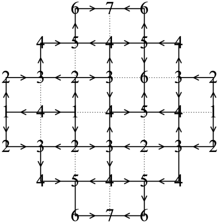
Types of boundary conditions

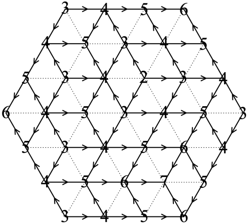
High entropy
b.c.





Domain wall
b.c.





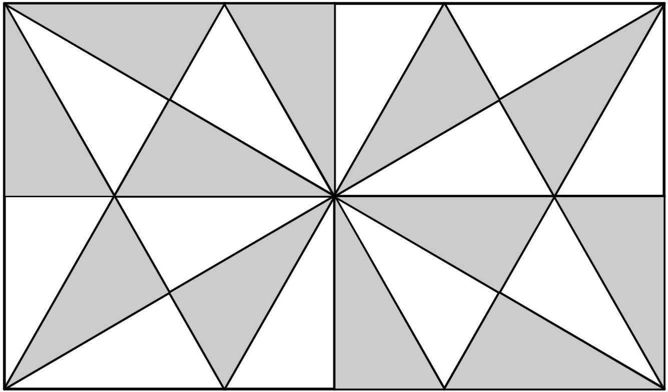
How to find tractable dimer-enumeration problems

(1) Use toroidal boundary conditions (or conditions on the boundary of a rectangle with the height function changing as little as possible) and apply Kasteleyn's methods.

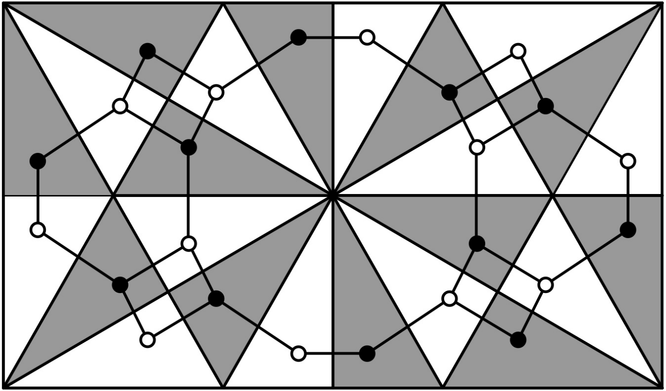
OR:

(2) Use domain-wall boundary conditions with the height function changing as drastically as possible and apply the other methods from the first two lectures.

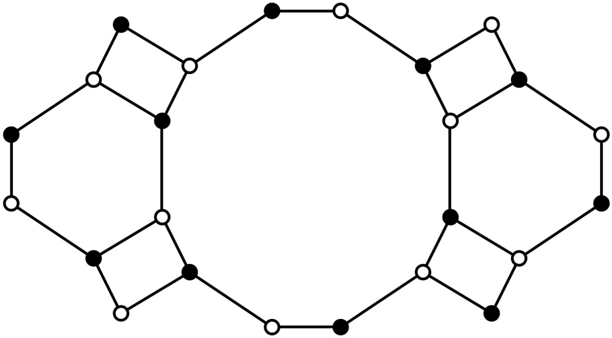
Example: Aztec dungeon



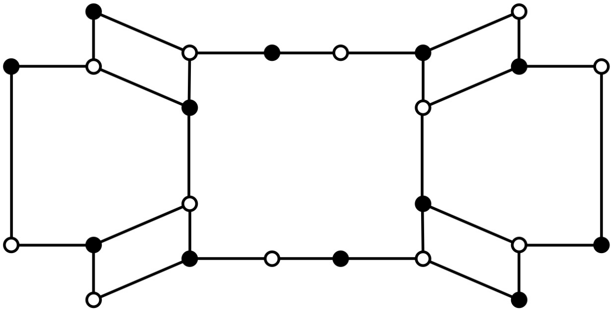
The Aztec dungeon



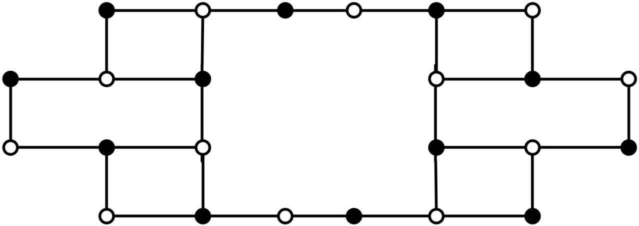
The Aztec dungeon



The Aztec dungeon



The Aztec dungeon



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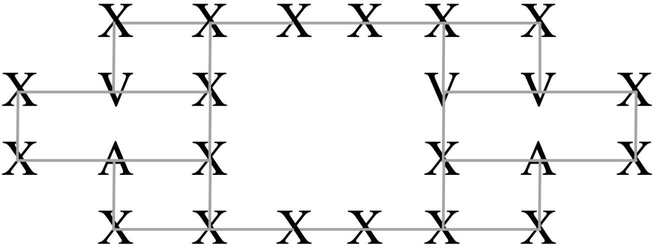


The Aztec dungeon

```
$ ./yaxmathematica
warning: this program uses gets(), which is unsafe.
 XXXXXX
XVX  XVX
XAX  XAX
 XXXXXX
x=-1; Abs[Det[{{x,1,0,0,1,0,0,0,0,0,0,0},
{0,x,1,0,0,0,0,0,0,0,0,0},
{0,0,x,0,0,1,0,0,0,0,0,0},
{1,0,0,x,1,0,0,0,0,0,0,0},
{0,0,1,0,0,1,0,x,0,0,0,0},
{0,0,0,0,0,x,0,0,x,0,0,0},
{0,0,0,x,0,0,1,0,0,0,0,0},
{0,0,0,0,x,0,x,0,0,x,0,0},
{0,0,0,0,0,0,0,x,1,0,0,x},
{0,0,0,0,0,0,x,0,0,1,0,0},
{0,0,0,0,0,0,0,0,0,x,1,0},
{0,0,0,0,0,0,0,x,0,0,x,1}}]]
$
```

The Aztec dungeon

```
In[29]:= x = -1;
         Abs[Det[{{x, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
             {0, x, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
             {0, 0, x, 0, 0, 1, 0, 0, 0, 0, 0, 0},
             {1, 0, 0, x, 1, 0, 0, 0, 0, 0, 0, 0},
             {0, 0, 1, 0, 0, 1, 0, x, 0, 0, 0, 0},
             {0, 0, 0, 0, 0, x, 0, 0, x, 0, 0, 0},
             {0, 0, 0, x, 0, 0, 1, 0, 0, 0, 0, 0},
             {0, 0, 0, 0, x, 0, x, 0, 0, x, 0, 0},
             {0, 0, 0, 0, 0, 0, 0, x, 1, 0, 0, x},
             {0, 0, 0, 0, 0, 0, x, 0, 0, 1, 0, 0},
             {0, 0, 0, 0, 0, 0, 0, 0, 0, x, 1, 0},
             {0, 0, 0, 0, 0, 0, 0, x, 0, 0, x, 1}}]]

Out[29]= 26
```

The Aztec dungeon

Ciucu (2002): The number of matchings of the Aztec dungeon graph is always a power of 13 or twice a power of 13.

Compare with Yang's unpublished 1991 result: The number of matchings of the fortess graph is always a power of 5 or twice a power of 5.

What's new?

Most of what I've talked about so far is several decades old.

If I were teaching a full-semester course for students contemplating research in combinatorics, I'd want to cover more recent developments.

Here are some articles I'd read to prepare myself to teach such a course (in addition to learning more about statistical mechanics and cluster algebras and integrable systems).

There are many more I'd want to read; this is just a sample!

1994

John Stembridge, Some hidden relations involving the ten symmetry classes of plane partitions

Symmetrical plane partitions correspond to symmetrical lozenge-tilings of hexagons.

Stembridge noticed (and proved) that in many cases, the number of tilings with a 2-fold symmetry equals the evaluation at $q = -1$ of the q-enumeration of the tilings, where a plane partition consisting of n cubes is assigned weight q^n.

This "$q = -1$ phenomenon" led the way to the discovery of the Cyclic Sieving Phenomenon by Reiner, Stanton, and White in 2004.

2000

Henry Cohn, 2-adic behavior of numbers of domino tilings

The number of domino tilings of a $2n$-by-$2n$ square was known to be of the form $2^n f(n)^2$; Cohn used Kasteleyn's formula to show that $f(n)$ is 2-adically continuous.

Similar results are known conjecturally, e.g., for domino tilings of the $2n$-by-$4n$ rectangle, and even domino tilings of the L-shaped region obtained from removing a $4n$-by-$4n$ square from the corner of an $8n$-by-$8n$ square.

Kasteleyn's method is unlikely to yield a proof for the L-shaped region; a different sort of explanation is probably required.

2001

Greg Kuperberg, Kasteleyn cokernels

The cokernel of the Kasteleyn matrix K of a graph G is an abelian group whose order is the number of matchings of the graph; it depends only on G (not on one's choice of K).

It is analogous to the critical group of a graph (the cokernel of the Laplacian), and indeed there are cases where the two notions coincide.

There are many fundamental open questions about the Kasteleyn cokernel.

2002

Rick Kenyon, The Laplacian and $\overline{\partial}$ operators on critical planar graphs

Kasteleyn's method of choosing edge-weights is global.

(Complex) Kasteleyn weights of the edges of a graph can be derived locally from an embedding of the graph in the complex plane provided that the embedding is "isoradial"; that is, vertices belonging to a face are concyclic, with all circles having the same radius.

Using isoradial embeddings opens the door to applying continuous and discrete analytic function theory to the study of the asymptotics of the dimer model.

2004

David Speyer, Perfect matchings and the octahedron recurrence

A common framework underlying much study of the dimer model is the octahedron recurrence

$$f(i,j;k+1)f(i,j;k-1) =$$

$$f(i-1,j;k)f(i+1,j;k)+$$

$$f(i,j-1;k)f(i,j+1;k)$$

describing a function on $\mathbb{Z}^3$.

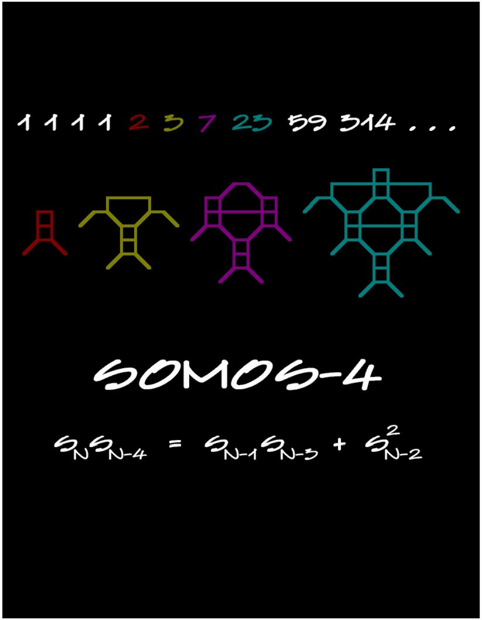
You've seen this with generalized shuffling and Kuo condensation.

2004

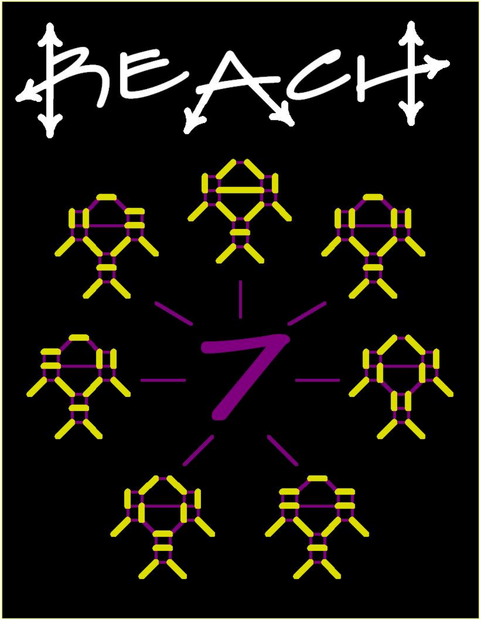
Speyer and others figured out how to reverse engineer dimer models from integer sequences satisfying one-dimensional octahedron recurrences, notably the Somos-4 sequence 1,1,1,1,2,3,7,23,59,314,. . . satisfying

$$S_n S_{n-4} = S_{n-1} S_{n-3} + S_{n-2} S_{n-2}$$

2004



2004



2007

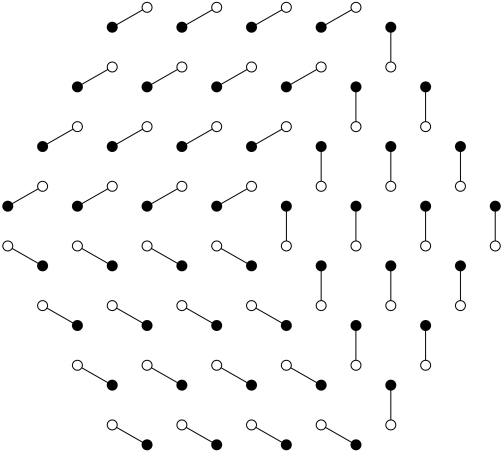
Benjamin Young, Computing a pyramid partition generating function with dimer shuffling

MacMahon's generating function

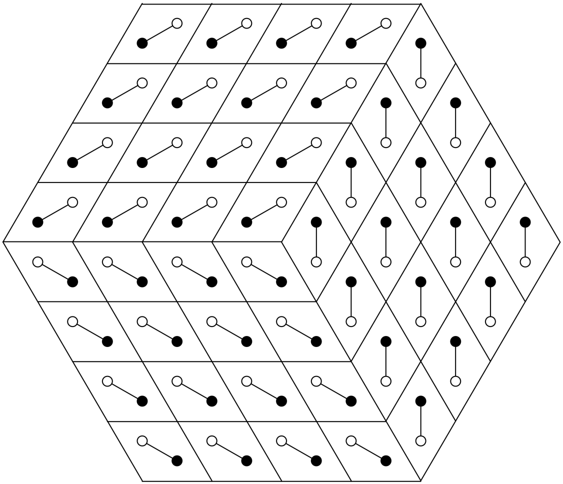
$$\frac{1}{(1-q^1)^1(1-q^2)^2(1-q^3)^3\cdots}$$

for unconstrained plane partitions can be seen as the partition function (in the stat mech sense) of a dimer model in the infinite hexagon lattice constrained "near infinity":

2007

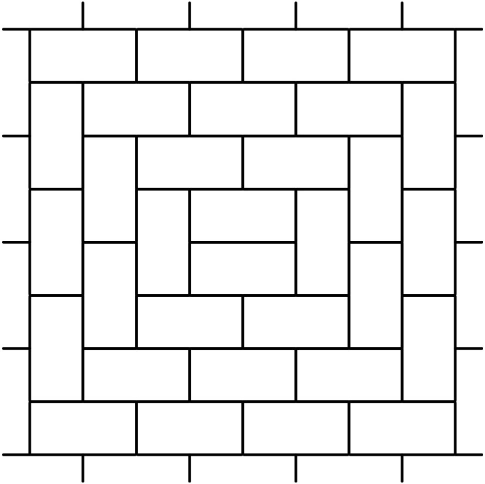


2007



2007

Kenyon and Szendroi had looked at the square-lattice version of this:



2007

Young proved a conjecture of Kenyon and Szendroi: if we weight finite "fillings" of the "square-lattice empty room" in the natural way (giving a dimer configuration weight q^n if it can be obtained from the the initial configuration using n face twists but no fewer) then the sum of the weights of all such configurations is

$$\frac{(1+q)^1(1+q^3)^3(1+q^5)^5\cdots}{(1-q^2)^2(1-q^4)^4(1-q^6)^6\cdots}$$

There are probably more such formulas awaiting discovery.

2008

Alexei Borodin and Vadim Gorin, Shuffling algorithm for boxed plane partitions

How often do you run into commuting stochastic matrices giving rise to two commuting dimensions of time evolution?!

2011

Eric Nordenstam and Benjamin Young, Domino shuffling on Novak half-hexagons and Aztec half-diamonds

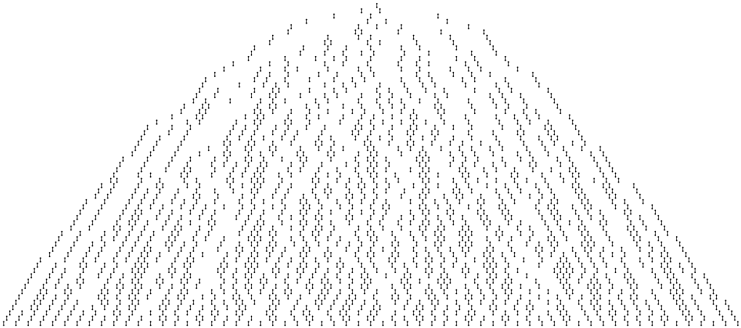


FIGURE 2. Order 100 half-hexagon, as an interlacing particle process

2014

Forest Tong, Generalizing the divisibility property of rectangle domino tilings

I had noticed years earlier (generalizing a fairly well-known property of the Hemachandra-Fibonacci numbers) that if $R(m, n)$ denotes the number of domino tilings of the m-by-n rectangle, then $R(m, n)$ divides $R(m', n')$ whenever $m + 1$ divides $m' + 1$ and $n + 1$ divides $n' + 1$.

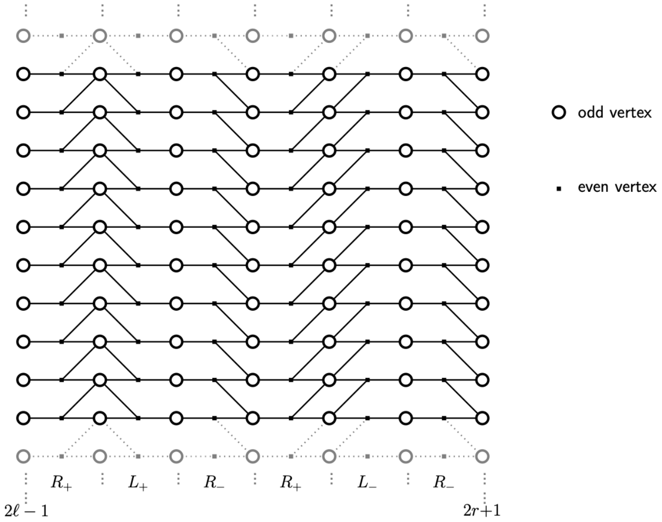
Tong gave a combinatorial proof.

2015

Cédric Boutillier, Jérémie Bouttier, Guillaume Chapuy, Sylvie Corteel, and Sanjay Ramassamy, Dimers on Rail Yard Graphs

The authors presented a new framework that unified many earlier results and simplified proofs of old results while making new ones possible.

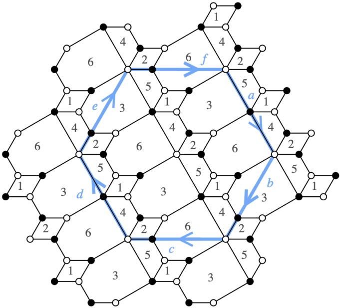
2015



2018

Tri Lai and Gregg Musiker, Dungeons and Dragons: Combinatorics for the dP_3 Quiver

"In this paper, we utilize the machinery of cluster algebras, quiver mutations, and brane tilings to study a variety of historical enumerative combinatorics questions all under one roof."



2019

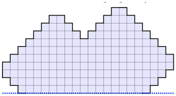
Grant Barkley and Ricky Liu, Channels, Billiards, and Perfect Matching 2-Divisibility

"We also establish a surprising connection between 2-divisibility of m_G and dynamical systems by showing an equivalency between channels and billiard paths."

2021

Mihai Ciucu, Cruciform regions and a conjecture of Di Francesco

Among other things, Ciucu shows that the number of domino tilings of "elbow" regions like



is given by the formula

$$2^{n(n+1)/2}n!\frac{H(2n+1)H(a)H(b)}{H(n+a+1)H(n+b+1)}$$

where $H(n) = 0!1!2!\cdots(n-1)!$.

(The picture shows the case $n = 7$, $a = 3$, $b = 4$.)

2023

Tomas Berggren and Alexei Borodin, Geometry of the doubly periodic Aztec dimer model

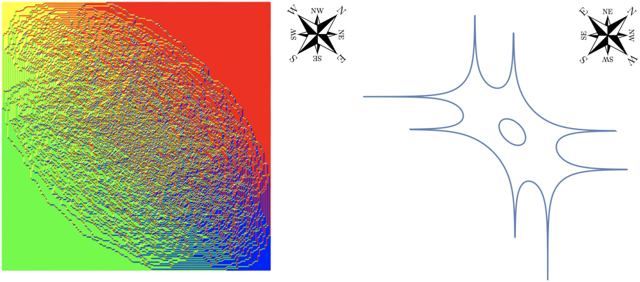
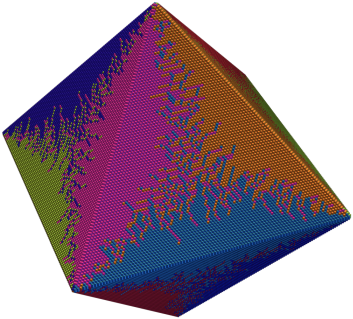


Figure 1: A randomly generated tiling (via so-called domino-shuffling, see Elkies–Kuperberg–Larsen–Propp [35], Jockusch-Propp-Shor [48] and Propp [82]) and the associated amoeba. The

2023

Nishant Chandgotia, Scott Sheffield, Catherine Wolfram,
Large deviations for the 3D dimer model



Bypassing exact combinatorial methods entirely, the authors made inroads on the hitherto untouchable topic of three-dimensional dimer models.

Quantum dimer model

I know nothing about this, but Roderich Moessner writes:

"Quantum dimer models have been influential in the quest for topological/fractionalised magnetic phases. One of the central ingredients here has been the observation that one can 'Rokhsar-Kivelsonize' classical dimer models, so that their correlations are those of a quantum model with a particular combination of diagonal and off-diagonal terms in the Hamiltonian."

It would be good to at least know what these words means, and why physicists find the quantum dimer model interesting.

Open enumerative problems

See

Tri Lai, https://arxiv.org/abs/2109.01466 Problems in the Enumeration of Tilings

and

James Propp, https://arxiv.org/abs/math/9904150 Enumeration of Matchings: Problems and Progress

The community

The listserv DOMINO@listserv.uml.edu is a watering-hole for people who are interested in this topic.

I also run ROBBINS@listserv.uml.edu (focusing on octahedron-like recurrences and integrable phenomena of a combinatorial nature) and DAC@listserv.uml.edu (focusing on dynamical algebraic combinatorics).

Thanks for listening!