Dimers, Webs, and Local Systems

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Abstract

For a planar bipartite graph G equipped with a $\mathrm{SL}(n)$-local system, we will discuss how the determinant of the associated Kasteleyn matrix counts "n-multiwebs" (generalizations of n-webs) in G, weighted by their web-traces. Time permitting, we will demonstrate how this fact can be used to study random n-multiwebs in graphs on some simple surfaces. These talks are based on joint work with Rick Kenyon and Haolin Shi [DKS22].

$$\{\text{Low Dimensional Geometry and Topology}\}$$
$$\bigcap$$
$$\{\text{Combinatorics}\}$$

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Special linear group

Let $\mathrm{SL}(n)$ denote the group of $n \times n$ matrices with determinant equal to 1, over the complex numbers.

Representation varieties

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$\mathrm{SL}(2)$ representation variety for the torus

$$\Gamma = \pi_1(\text{torus}) = \langle x, y; xy = yx \rangle ,$$

$$\mathrm{SL}(2) \text{ rep variety} = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in \mathrm{M}(2)^2 \cong \mathbb{C}^8; \right.$$

$$\left. \begin{matrix} ad - bc = 1 & AD - BC = 1 \\ aA + bC = Aa + Bc & aB + bD = Ab + Bd \\ cA + dC = Ca + Dc & cB + dD = Cb + Dd \end{matrix} \right\}.$$

Character varieties

One way to obtain such a representation is to take the monodromy representation associated with a geometric structure, such as the monodromy of a flat $\mathrm{SL}(n)$ connection on a surface. Since monodromy representations tend to be defined only up to conjugation, we would like to identify these representations up to conjugation.

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However, the quotient space (representation variety)$/\sim$ by conjugation can be pathological, for example one-point sets might not be closed–in particular, the quotient can be non-Hausdorff.

Example

Let $\Gamma = \langle x \rangle \cong \mathbb{Z}$, let $\rho_t(x) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and let $\rho_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Then $\rho_t \to \rho_0$ as $t \to 0$, but $\rho_t \sim \rho_1$ for all $t \neq 0$ while ρ_1 and ρ_0 are not conjugate. Note though that $\mathrm{Tr}\,\rho_t = \mathrm{Tr}\,\rho_0$ for all t.

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It turns out that this quotient space, called the **character variety**, also has the structure of an affine algebraic set.

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Trace functions

We take the algebraic geometric approach and study the character variety χ by means of its algebra $\mathbb{C}[\chi]$ of regular–i.e. polynomial–functions. An important family of regular functions is defined as follows: for each $\gamma \in \Gamma$, define the **trace function** $\mathrm{Tr}_\gamma \in \mathbb{C}[\chi]$ by

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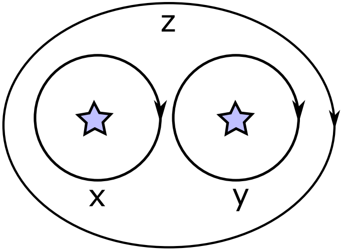
A theorem of classical invariant theory, due to Procesi [Pro76], implies that the trace functions $\{\mathrm{Tr}_{\gamma}; \gamma \in \Gamma\}$ generate the algebra $\mathbb{C}[\chi]$ of regular functions as an algebra. In other words, every invariant regular function on the representation variety is a polynomial in trace functions.

Topological setting

We will now focus on the case where $\Gamma = \pi_1(S)$ is the fundamental group of a finite-type surface S.

Example: $\mathrm{SL}(2)$ character variety for the pair of pants

Let $\Gamma = \pi_1(\text{pair of pants}) = \langle x \rangle * \langle y \rangle \cong \mathbb{Z} * \mathbb{Z}$. By Procesi's theorem, the algebra of $\mathbb{C}[\chi]$ of regular functions on the $\mathrm{SL}(n)$ character variety is generated by the trace functions $\{\mathrm{Tr}_\gamma; \gamma \text{ word in } x, x^{-1}, y, y^{-1}\}$.



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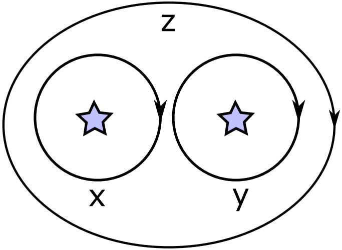
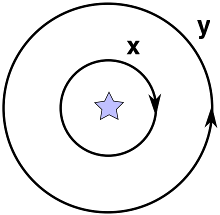


In the case $n = 2$, as an algebra $\mathbb{C}[\chi]$ is isomorphic to $\mathbb{C}[x, y, z]$–the polynomial algebra in three variables–by the map $x \mapsto \mathrm{Tr}_x$, $y \mapsto \mathrm{Tr}_y$, and $z \mapsto \mathrm{Tr}_{xy}$.

Example: $\mathrm{SL}(3)$ character variety for the annulus

Let $\Gamma = \pi_1(\text{annulus}) = \langle x \rangle \cong \mathbb{Z}$. By Procesi's theorem, the algebra $\mathbb{C}[\chi]$ of regular functions on the $\mathrm{SL}(n)$ character variety is generated by the trace functions $\{\mathrm{Tr}_{x^k}; k \in \mathbb{Z}\}$.



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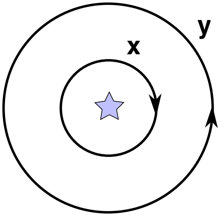

In the case $n = 3$, as an algebra $\mathbb{C}[\chi]$ is isomorphic to $\mathbb{C}[x, y]$–the polynomial algebra in two variables–by the map $x \mapsto \mathrm{Tr}_x$ and $y \mapsto \mathrm{Tr}_{x^{-1}}$.[2]

[2] In the case $n = 2$, $\mathbb{C}[\chi] \cong \mathbb{C}[x]$ since $\mathrm{Tr}(A) = \mathrm{Tr}(A^{-1})$ for all $A \in \mathrm{SL}(2)$.

Relations among trace functions: $n = 2$

Besides the trivial trace relation $\mathrm{Tr}\, Id = n$, it turns out that essentially the only nontrivial relation among $\mathrm{SL}(n)$ trace functions comes from the Cayley-Hamilton theorem, which says that square matrices satisfy their own characteristic equation.

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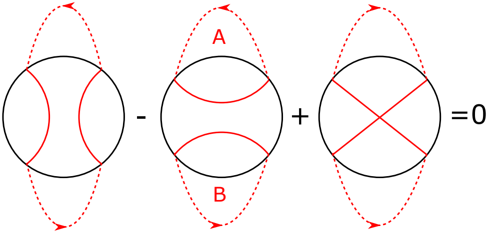

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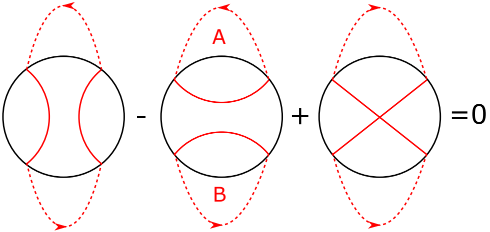

These kinds of pictorial relations are called **skein relations**. One of their trademarks is that they are purely local relations, that is, they take place in a contractible disk.[3]

[3] In the 3-dimensional setting, skein relations can also be used to define the famous Jones polynomial of a knot.

Relations among trace functions: $n = n$

For $n > 2$, the relations are more complicated to depict graphically. In the 90s, it was realized that a good pictorial way to think about these relations is via graphs [Kup96, Sik01].

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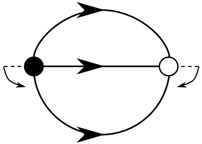
Definition

A $\mathrm{SL}(n)$ **web** is an oriented n-valent graph in the surface such that each vertex is either a sink or a source–meaning n edges all going in or out, respectively.[a][b] Webs are considered up to homotopy in the surface.

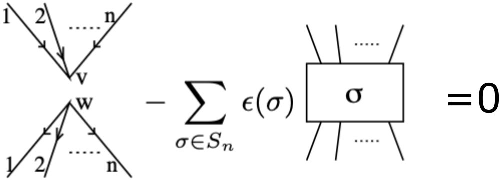
[a] We also choose a linear ordering of the incident half-edges at each vertex.

[b] Note that a web is just an n-valent bipartite (unoriented) graph, with such linear orderings.



Relations among trace functions: $n = n$

As we will see more concretely later, it makes sense to talk about the **web trace** function $\mathrm{Tr}_W \in \mathbb{C}[\chi]$ on the character variety associated to a web W. The following determinant-like skein relation essentially captures all nontrivial relations among SL_n trace functions.[4]



[4] Picture courtesy of Adam Sikora, see [Sik01].

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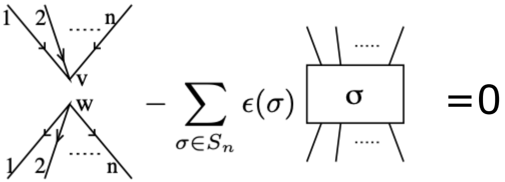

In the remaining sections, we will discuss an application of web traces to combinatorics, in particular Kasteleyn theory. Before that, let's give one more example.

[4] Picture courtesy of Adam Sikora, see [Sik01].

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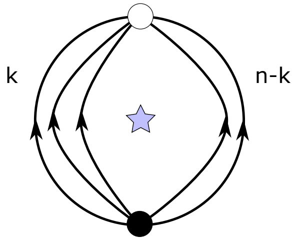

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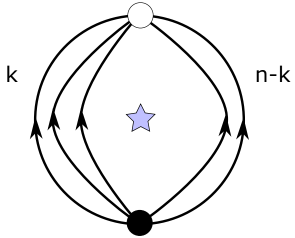

Generalizing the $n = 1$ and $n = 2$ cases before, as an algebra $\mathbb{C}[\chi]$ is isomorphic to $\mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$, the polynomial algebra in $n - 1$ variables.

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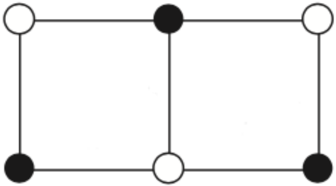
Setting: planar bipartite graphs

For the next two sections, $\mathcal{G} = (V, E)$ will be a planar bipartite graph (together with a chosen embedding in the plane).[5]



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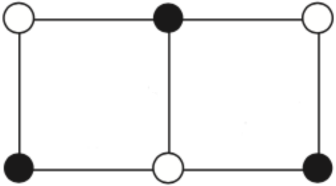


The vertex set $V = B \cup W$ is divided into black and white vertices. We always assume that there are the same number of black and white vertices, $|B| = |W|$. We always think of the edges as oriented from black to white.

[5]For simplicity, let us assume there is at most one edge between each pair of vertices.

Dimer covers

A **dimer cover**, or **perfect matching**, of the graph $\mathcal{G}$ is a collection of edges such that every vertex is contained in exactly one edge.

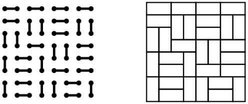


Figure: Domino tiling of the square grid

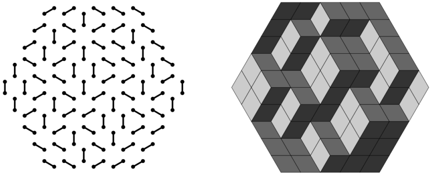
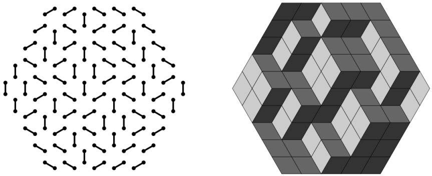


Figure: Lozenge tiling of the honeycomb

Dimer covers



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How many dimer covers does $\mathcal{G}$ have? Let Z_d denote the answer, called the **partition function**.[a]

[a] Note that, despite its name, Z_d is just a constant, not a function.

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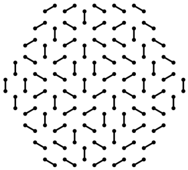

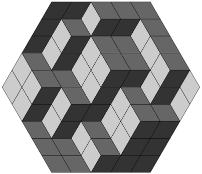

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One motivating problem is to study the uniform probability measure on the set of all dimer covers of $\mathcal{G}$, that is, Probability$(m) = 1/Z_d$ for each dimer cover m.

Kasteleyn matrix

The answer to the previous question is that the number Z_d of dimer covers of $\mathcal{G}$ can be computed as the determinant of a matrix K, the **Kasteleyn matrix**, which is just a signed adjacency matrix. More precisely, K is the $|W| \times |B|$ matrix defined by[6]

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Here, the $\varepsilon_{wb} \in \{\pm 1\}$ are signs following the **Kasteleyn rule**, namely that for each face of $\mathcal{G}$ of length k the product of the signs on the edges around the face is equal to $(-1)^{k/2+1}$. Such signs always exist, but are not unique.

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Theorem (Kasteleyn, Temperley–Fisher, 1963)

$$Z_d = |\det K| \, .$$

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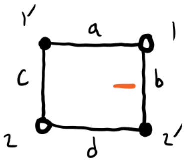
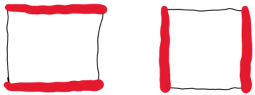
Example





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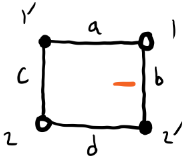
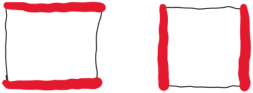
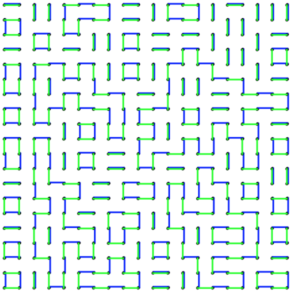




$$K = (K_{ii'}) = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}, \quad \det K = ad + bc.^{7}$$

[7] Here we are demonstrating the version of Kasteleyn's theorem with edge weights, where the weight of each dimer cover m is the product of its edge weights, $\prod_e m_e$.

Sketch of proof of Kasteleyn's theorem



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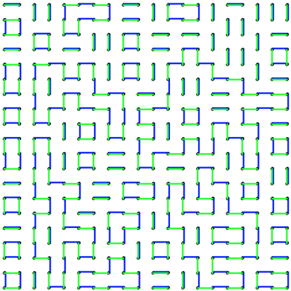

Proposition

Given a cycle of length $2k$ enclosing ℓ points, the alternating product of signs around this cycle is $(-1)^{1+k+\ell}$.

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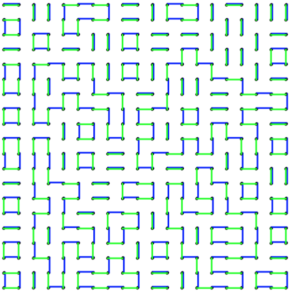

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See, e.g., Kenyon's notes on dimers, [Ken09].

An application of Kasteleyn's theorem

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Corollary

Given a set of edges $X = \{w_1 b_1, \ldots, w_k b_k\}$, *the probability that all edges in* X *occur in a dimer cover is*

$$\left(\prod_{i=1}^{k} K(b_i, w_i)\right) \det \left(K^{-1}(w_i, b_j)\right)_{1 \le i,j \le k}.$$

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The proof uses the Jacobi Lemma that says that a minor of a matrix A is $\det A$ times the complementary minor of A^{-1}.

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Double dimer covers

A **double dimer cover** of $\mathcal{G}$ is the result of overlaying two dimer covers, forgetting the order in which they are overlayed. In other words, a double dimer cover is a collection of loops and doubled edges covering all vertices.

Double dimer covers

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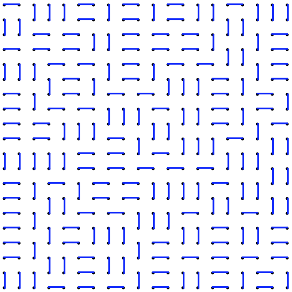

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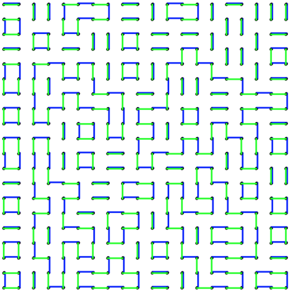

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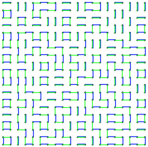

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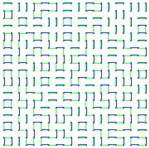

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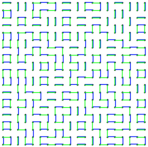

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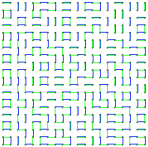

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$$\text{Probability}(m) = 2^{\#(\text{loops in } m)} / Z_{2d}.$$

SL(2) connections

Attach a copy of $\mathbb{C}^2$ to each vertex of $\mathcal{G}$. A $\mathrm{SL}(2)$ **connection** $\Phi = \{\phi_{bw}\}$ is the assignment of a matrix $\phi_{bw} \in \mathrm{SL}(2)$ to each edge bw, oriented from black to white. So we are thinking of $\phi_{bw} : \mathbb{C}^2 \to \mathbb{C}^2$ where the domain is at the black vertex, and the codomain is at the white vertex. We also put $\phi_{wb} = \phi_{bw}^{-1}$. If $\gamma = b_1 w_1 b_2 w_2 \cdots b_k w_k b_1$ is an oriented loop in $\mathcal{G}$, the **monodromy** around γ with respect to the connection Φ is

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Note since $\mathrm{Tr}\, A = \mathrm{Tr}\, A^{-1}$ for all $A \in \mathrm{SL}(2)$, it follows that $\mathrm{Tr}\, Mon(\gamma)$ is independent of the choice of orientation of γ. Also note that if $\phi_{bw} = Id$ is the identity matrix for all edges–this is called the **identity connection**–then $\mathrm{Tr}\, Mon(\gamma, Id) = \mathrm{Tr}\, Id = 2$ for each loop γ.

Kasteleyn matrix: $n = 2$

The Kastelyn matrix for $\mathcal{G}$ in the presence of a $\mathrm{SL}(2)$ connection $\Phi = \{\phi_{bw}\}$ is a straightforward generalization of the single dimer version. First, we define a $|W| \times |B|$ matrix K with 2×2 matrix-valued entries by

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Theorem (Kenyon, 2014)

$$\det \tilde{K} = \sum_{m \in \Omega_2} \prod_{\gamma \text{ loop in } m} \mathrm{Tr}\, Mon(\gamma, \Phi).$$

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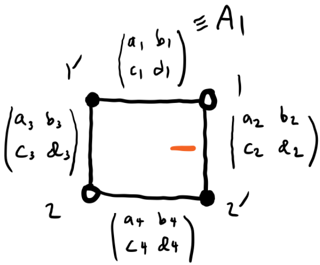
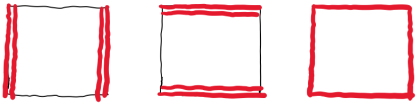
In particular, for $\Phi = Id$ the identity connection, $\det \tilde{K} = Z_{2d}$.
Kenyon's proof took advantage of the symmetries of $\mathrm{SL}(2)$. Later on, we will give a more elementary proof.

Example





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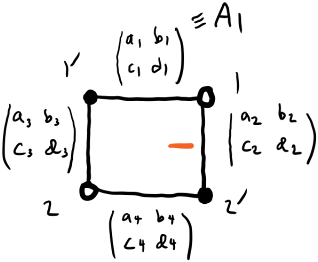
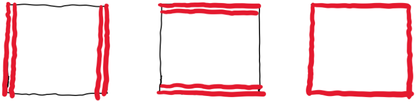
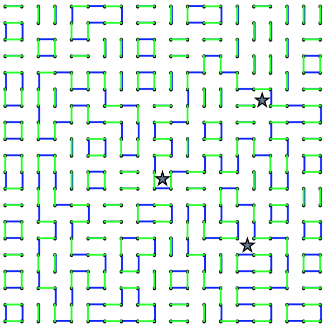




$$K = (K_{ii'}) = \begin{pmatrix} A_1 & -A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & -a_2 & -b_2 \\ c_1 & d_1 & -c_2 & -d_2 \\ a_3 & b_3 & a_4 & b_4 \\ c_3 & d_3 & c_4 & d_4 \end{pmatrix},$$

$$\det K = 1 + 1 + \operatorname{Tr} A_3^{-1} A_4 A_2^{-1} A_1 = 1 + 1 + \operatorname{Tr} A_1^{-1} A_2 A_4^{-1} A_3.$$

Application of the $\mathrm{SL}(2)$ Kasteleyn theorem



The $\mathrm{SL}(2)$ Kasteleyn theorem has been used to show that the probabilities for lamination types of double dimer covers in the grid in the punctured plane converge and are conformally invariant in the scaling limit as the mesh size of the grid goes to zero [Ken14, Dub19, BC21]. Our goal in the next section is to generalize the $\mathrm{SL}(2)$ Kasteleyn theorem to $\mathrm{SL}(n)$.

Table of Contents

Multiwebs

Throughout fix $n \geq 1$. Let $\mathcal{G}$ be a bipartite planar graph as before.

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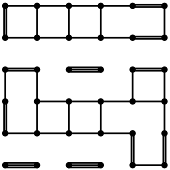
Definition

A **multiweb** m is the result of overlaying n dimer covers of $\mathcal{G}$, forgetting the order in which they were laid. More formally, it is a function $m : E \to \{0, 1, 2, \ldots, n\}$ such that each vertex has n incident edges counting multiplicity, that is, $\sum_{v \sim u} m(vu) = n$. [a] [b] Denote the set of multiwebs by $\Omega_n = \Omega_n(\mathcal{G})$.

[a] For bipartite graphs these notions coincide since every n-valent bipartite graph has a dimer cover.

[b] Multiwebs are called "weblike subgraphs" by Fraser-Lam-Le [FLL19].



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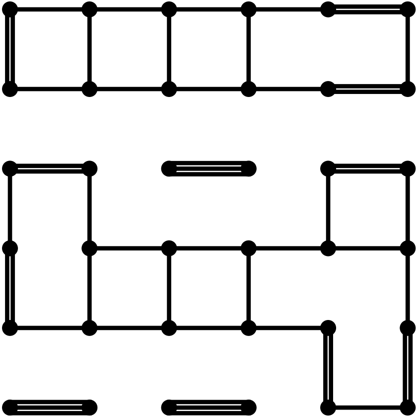


Graph connections

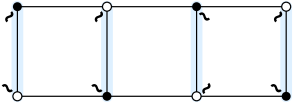
A $\mathrm{SL}(n)$ **connection** $\Phi = \{\phi_{bw}\}$ on $\mathcal{G}$ is defined just as a $\mathrm{SL}(2)$ connection, namely an assignment of a matrix $\phi_{bw} \in \mathrm{SL}(n)$ to each edge bw. In fact, our main result will be valid for any $\mathrm{M}(n)$ **connection**, where the ϕ_{bw} are any, not necessarily invertible, matrices.

Traces for webs?

The next step is to define the trace $\mathrm{Tr}\,(m, \Phi)$ of a multiweb $m \in \Omega_n$ with respect to a connection Φ. In the case $n = 2$, a nontrivial 2-web was essentially a loop, so we could just take the trace of the monodromy of the connection around the loop. However, for webs the monodromy is not defined. Rather, the traces will be constructed via a **tensor network construction**.

Technical aspect: cilia data

One technical aspect is that in order to define the web trace, we need to choose **cilia data**, consisting of one cilium per vertex of $\mathcal{G}$.



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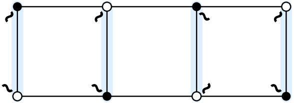


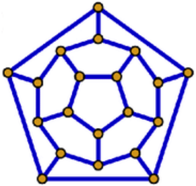
More precisely, at each vertex v, the orientation of the surface determines a cyclic ordering of the half-edges incident to that vertex[9], but we require a linear ordering. The choice of cilium at that vertex upgrades the cyclic order to a linear order by telling us which half-edge at v appears first in the order. We will denote this cilia data by L.[10]

[9] Our convention is to choose the counterlockwise orientation around black vertices, and the clockwise orientation around white vertices.

[10] For another setting where cilia appear, see the work of Fock–Rosly [FR99].

Proper multiwebs

A multiweb $m \in \Omega_n$ is a **proper multiweb** if it contains no edges of multiplicity > 1. More formally, $m : E \to \{0, 1, 2, \ldots, n\}$ takes only the values 0 and 1. In other words, a proper multiweb is an n-valent spanning subgraph of $\mathcal{G}$.[11]



[11] Figure taken from [KFSH20].

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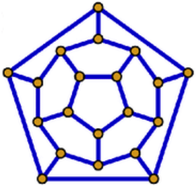
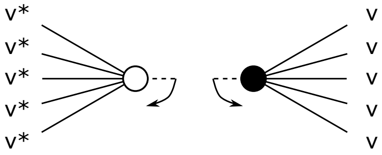


As a technical point, at each vertex v the linear order of half-edges of $\mathcal{G}$ at v, determined by the cilia data L, restricts to a linear order of the n half-edges of m at v, which we also denote by L.

[11] Figure taken from [KFSH20].

Trace of a proper multiweb

Fix a $\mathrm{M}(n)$ connection Φ, a cilia data L, and a proper multiweb m. We assign to this data a number $\mathrm{Tr}_L(m, \Phi)$ as follows. This is a standard tensor network definition, see e.g. [Sik01]. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of $V = \mathbb{C}^n$, and let $\{y_1, y_2, \ldots, y_n\}$ be the corresponding dual basis in V^*. Attach a copy of V to each half-edge incident to each black vertex b, and attach a copy of its dual V^* to each half-edge incident to each white vertex w.



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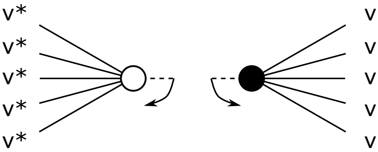

For each black vertex b, assign the **codeterminant**

$$v_b = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \quad \in V \otimes V \otimes \cdots \otimes V.$$

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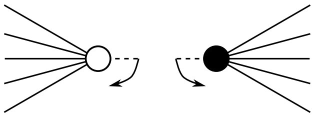

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Here, the order of the tensor factors is determined by the linear order at the vertex b, as indicated by the choice of cilia. Similarly, for each white vertex w define the dual codeterminant

$$u_w = \sum_{\sigma \in S_n} (-1)^{\sigma} y_{\sigma(1)} \otimes y_{\sigma(2)} \otimes \cdots \otimes y_{\sigma(n)} \quad \in V^* \otimes V^* \otimes \cdots \otimes V^*.$$

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The trace $\mathrm{Tr}_L(m, \Phi)$ is defined by tensoring the codeterminants at all the blacks, applying the connections ϕ_{bw} along all the edges *bw* from black to white, then contracting the resulting tensor with the dual codeterminants taken at all the whites:[12]

[12] The web trace is independent of the choice of basis for V, because the codeterminants transform by the determinant of the change of basis matrix, while the dual codeterminants transform by the inverse of the determinant of the change of basis matrix.

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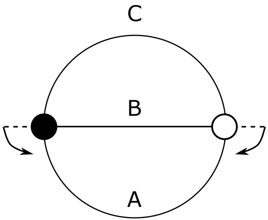
Example: theta graph for $\mathrm{SL}(3)$ and $\mathrm{M}(3)$ connections



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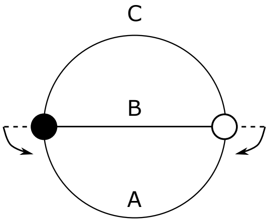

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$$v_{\text{black}} = x_r \otimes x_g \otimes x_b - x_r \otimes x_b \otimes x_g + \cdots - x_b \otimes x_g \otimes x_r$$

and

$$u_{\text{white}} = y_r \otimes y_g \otimes y_b - y_r \otimes y_b \otimes y_g + \cdots - y_b \otimes y_g \otimes y_r.$$

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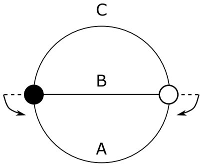

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The contraction contains 36 terms; for example, contracting the first terms of v_{black} and u_{white} gives $A_{rr}B_{gg}C_{bb}$, and contracting the first term of v_{black} and the second term of u_{white} gives $-A_{rr}B_{bg}C_{gb}$.

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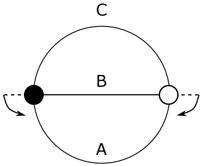

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$$\mathrm{Tr}\,(m) = \mathrm{Tr}\,(AB^{-1})\mathrm{Tr}\,(CB^{-1}) - \mathrm{Tr}\,(AB^{-1}CB^{-1})$$

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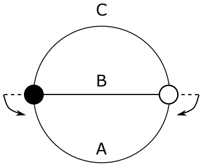

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$$u_{\text{white}} = y_r \otimes y_g \otimes y_b - y_r \otimes y_b \otimes y_g + \cdots - y_b \otimes y_g \otimes y_r.$$

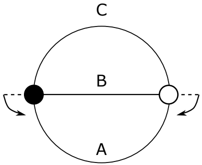
The contraction contains 36 terms; for example, contracting the first terms of v_{black} and u_{white} gives $A_{rr}B_{gg}C_{bb}$, and contracting the first term of v_{black} and the second term of u_{white} gives $-A_{rr}B_{bg}C_{gb}$. Summing, the trace can be written compactly as

$$\mathrm{Tr}\,(m) = \mathrm{Tr}\,(AB^{-1})\mathrm{Tr}\,(CB^{-1}) - \mathrm{Tr}\,(AB^{-1}CB^{-1})$$

or, more symmetrically, as

$$[xyz]\,\det(xA + yB + zC),$$

Example: theta graph for $\mathrm{SL}(3)$ and $\mathrm{M}(3)$ connections



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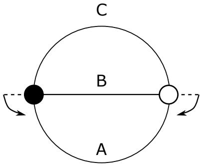

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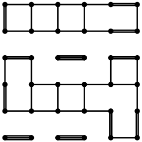
It turns out that this last equation is valid for any $\mathrm{M}(3)$ connection as well.

Trace of a multiweb

The definition of the trace for a general multiweb $m \in \Omega_n(\mathcal{G})$–where there can be arbitrary edge multiplicities m_e–is in terms of the previous definition of the trace for proper multiwebs.



13

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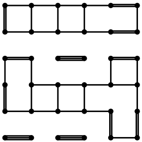
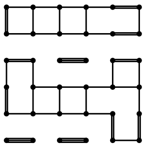


First, "split" each edge $e = bw$ of m of multiplicity $m_e = k$ into k parallel edges, and put the same connection ϕ_{bw} on each of these newly formed edges. Doing this for each edge, we naturally obtain a proper multiweb $\tilde{m}$[13] with a connection $\tilde{\Phi}$. Also, $\tilde{m}$ naturally acquires a cilia data $\tilde{L}$ from the cilia data L for m.

[13]...proper multiweb $\tilde{m} \in \Omega_n(\tilde{\mathcal{G}})$ in the associated "split" graph $\tilde{\mathcal{G}}$...

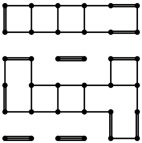
Trace of a multiweb



The trace is then defined by

$$\mathrm{Tr}_L(m, \Phi) = \frac{\mathrm{Tr}_{\tilde{L}}(\tilde{m}, \tilde{\Phi})}{\prod_e (m_e)!}.$$

Trace of a multiweb



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In other words, we divide the trace of the "split" proper multiweb $\tilde{m}$ by the factorials of the multiplicities m_e of the edges of the original multiweb m.

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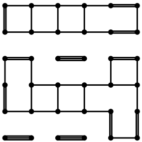

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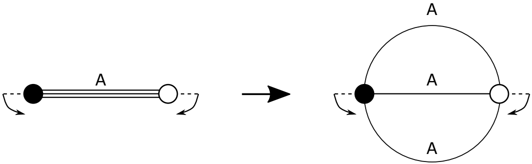
The trace of a multiweb is a mysterious quantity that is difficult to compute. In a moment, we will give a determinantal algorithm, using Kasteleyn theory, to compute web traces–albeit in exponential time. But first we give some more examples.

Example: trivial multiweb for $\mathrm{M}(n)$ and $\mathrm{SL}(n)$ connections

Consider the graph with two vertices and one edge, equipped with a $\mathrm{M}(n)$ connection, namely a $n \times n$ matrix A on the edge. This has a single multiweb m, which has weight n on the edge. Let $\tilde{m}$ be the associated "split" proper multiweb, with the matrix A on each edge. Shown below is the example $n = 3$.



14

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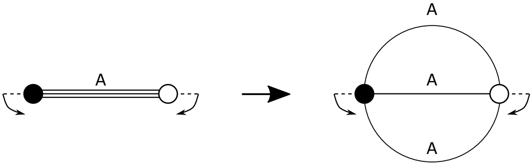

$$\mathrm{Tr}\, m(A) = \frac{\mathrm{Tr}\, \tilde{m}(A)}{n!} = \frac{[x_1 x_2 \cdots x_n] \det (x_1 A + x_2 A + \cdots + x_n A)}{n!} = \frac{n! \det A}{n!} = \det A.$$

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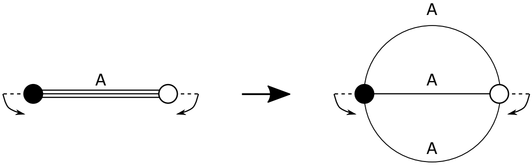

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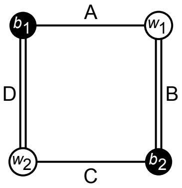
In particular, when $A \in \mathrm{SL}(n)$ the web trace of the trivial multiweb is 1, as we saw in the $n = 2$ case.[14]

[14] And the $n = 1$ case!

Example: loop for $\mathrm{M}(n)$ and $\mathrm{SL}(n)$ connections

Consider the cycle $b_1 w_1 b_2 w_2 \dots b_\ell w_\ell$ with edges $b_i w_i$ of multiplicity k, and edges $b_i w_{i-1}$ of multiplicity $n-k$, and an $\mathrm{M}(n)$-connection with parallel transport A_i from b_i to w_i, and B_{i-1} from b_i to w_{i-1}.



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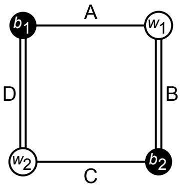

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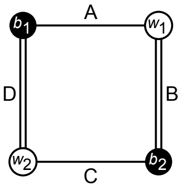
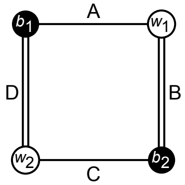


Then, we have $\mathrm{Tr}_L(m) = \pm\mathrm{Tr}(C)$, where C is the matrix product

$$C = (\widetilde{\wedge^{n-k} B_\ell})(\wedge^k A_\ell) \cdots (\widetilde{\wedge^{n-k} B_2})(\wedge^k A_2)(\widetilde{\wedge^{n-k} B_1})(\wedge^k A_1)$$

where the matrix factors are essentially the matrices of the induced maps on exterior products.

Example: loop for $\mathrm{M}(n)$ and $\mathrm{SL}(n)$ connections



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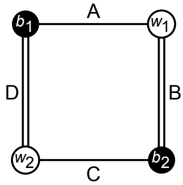

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For an $\mathrm{SL}(3)$-connection, and for the multiweb above, this becomes

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namely the trace of the monodromy when the cycle is oriented clockwise in the figure.

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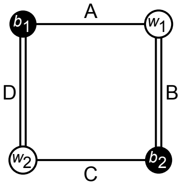

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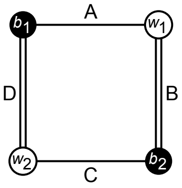
In particular, note that even though the cycle is not naturally oriented, the 3-web-trace nevertheless picks out an orientation: the one determined by following from black to white along the non-doubled edges.

Example: loop for $\mathrm{M}(n)$ and $\mathrm{SL}(n)$ connections



Lastly, for an $\mathrm{SL}(n)$-connection with total monodromy $M = B_\ell^{-1} A_\ell \cdots B_1^{-1} A_1$ clockwise around the loop, we have

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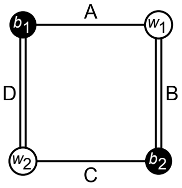

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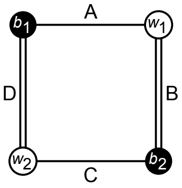

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which, in particular, is $\pm \mathrm{Tr}\,(M)$ when $k = 1$, i.e. the trace of the clockwise monodromy of the $\mathrm{SL}(n)$ connection around the loop.

Kasteleyn matrix: $n = n$

The Kasteleyn matrix $\tilde{K}$ associated to a $\mathrm{M}(n)$ connection on $\mathcal{G}$ is defined as for $\mathrm{SL}(2)$ connections, that is, it is the $n|W| \times n|B|$ matrix obtained by plugging the connection matrices ϕ_{bw} into the signed adjacency matrix for $\mathcal{G}$. The Kasteleyn matrix $\tilde{K}$ depends on the choice of Kasteleyn signs, as well as the choice of orderings of black and white vertices.

Kasteleyn matrix: $n = n$

Theorem (D.-Kenyon-Shi, 2022)

Let Φ be a $\mathrm{M}(n)$ connection on $\mathcal{G}$.

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- *For n odd, the web traces $\mathrm{Tr}_L(m, \Phi) = \mathrm{Tr}(m, \Phi)$ do not depend on the choice of cilia data, and*

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- *For n even, there exist explicit choices of cilia data $L = L_+$ for $\mathcal{G}$, called* **positive cilia data**, *such that*

$$\det \tilde{K}(\Phi) = \sum_{m \in \Omega_n} \mathrm{Tr}\,_{L_+}(m, \Phi).$$

In particular, in this case $\det \tilde{K}(\Phi)$ *does not depend on choices.*

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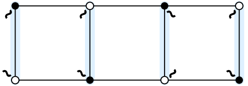
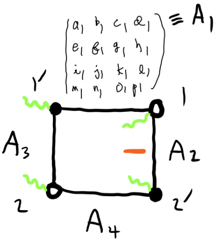
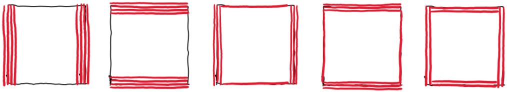
Note that the choice of positive cilia data L_+ is not unique. To choose one, take the cilia such that each face has an even number of cilia pointing in. To do this, pick a dimer cover of $\mathcal{G}$, and let the two cilia on each dimer point into the same face. Also, $\mathrm{Tr}_{L_+}(m, \Phi) = \mathrm{Tr}_{L'_+}(m, \Phi)$ for any positive cilia L_+ and L'_+.



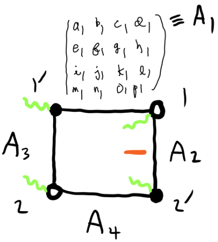
Example

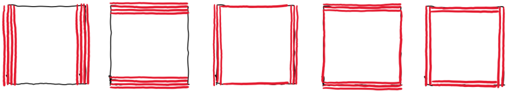




15 or $+\mathrm{Tr}\,(\widetilde{\Lambda^2 A_1})(\Lambda^2 A_2)(\widetilde{\Lambda^2 A_4})(\Lambda^2 A_3)$.

Example





$$K = (K_{ii'}) = \begin{pmatrix} A_1 & -A_2 \\ A_3 & A_4 \end{pmatrix},$$

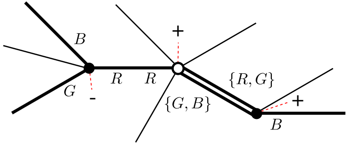
$$\pm \det K = 1 + 1 + \operatorname{Tr} A_3^{-1} A_4 A_2^{-1} A_1 + \operatorname{Tr} A_1^{-1} A_2 A_4^{-1} A_3 +$$

$$+ \operatorname{Tr} (\widetilde{\Lambda^2 A_3})(\Lambda^2 A_4)(\widetilde{\Lambda^2 A_2})(\Lambda^2 A_1).$$

[15] or $+\operatorname{Tr} (\widetilde{\Lambda^2 A_1})(\Lambda^2 A_2)(\widetilde{\Lambda^2 A_4})(\Lambda^2 A_3)$.

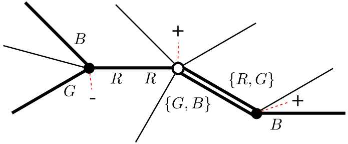
Sketch of proof: combinatorial interpretation of web traces

Denote by $\mathcal{C} = \{1, 2, \dots, n\}$ the set of n colors. A **(half-edge) coloring** of a multiweb $m \in \Omega_n$ assigns to each half-edge h a subset of colors of size m_{e_h}, such that all n colors are present at each vertex. More formally, a coloring is a function $c : \{\text{half edges}\} \to 2^{\mathcal{C}}$ such that $|c(h)| = m_{e_h}$ and $\bigcup_{\text{half edges } h \text{ incident to } v} c(h) = \mathcal{C}$ for all vertices v.



Given a coloring c, to each vertex v there is determined a local sign $c_v = c_v(L)$ by comparing the colors of the coloring to the colors taken in their natural order, starting as indicated by the cilia. Here, colors appearing in subsets on edges of multiplicity ≥ 2 are also taken in their natural order.

Sketch of proof: combinatorial interpretation of web traces

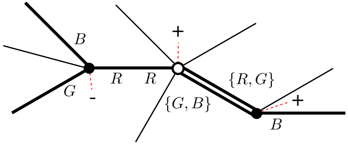


Proposition (determinantal formula for web traces)

$$\mathrm{Tr}_L(m, \Phi) = \sum_{colorings\ c} \prod_v c_v \prod_{e=bw} \det(\phi_{bw})_{S_e, T_e}.$$

Here, the determinants on the right hand side are the minors of the connection matrices ϕ_{bw} depending on the two subsets of colors on each edge $e = bw$, one set of colors T_e on the half-edge incident to the black vertex and one set of colors S_e on the half-edge incident to the white vertex. The proof is essentially a reformulation of the tensor network definition.

Sketch of proof: combinatorial interpretation of web traces



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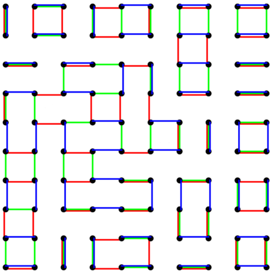
Corollary

For n odd, the web traces $\mathrm{Tr}_L(m, \Phi) = \mathrm{Tr}(m, \Phi)$ *do not depend on the choice of cilia data.*

This is because the signs c_v don't depend on the cilia, as the sign of the shift permutation $(23 \cdots n1)$ is $+1$ for n odd.

Sketch of proof: edge-n-colorings

If the connection Φ is diagonal, namely all the matrices ϕ_{bw} are diagonal, then the only colorings c that contribute to the trace $\mathrm{Tr}_L(m, \Phi)$ in the previous proposition are the **edge-n-colorings**, meaning for each edge $e = bw$ the color subsets of the two half-edges coincide. In other words, the edges are colored, rather than the half-edges. Note that an edge-n-coloring for m is nothing more than n dimer covers for m laid down in a particular order.[16]



[16]Since every n-valent bipartite graph has a dimer cover, every multiweb m has an edge-n-coloring, by induction.

Sketch of proof: edge-n-colorings

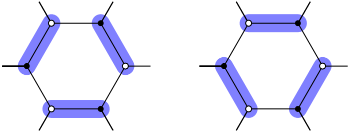
Proposition (planar web trace formula for diagonal connections)

For planar graphs and diagonal connections,

$$\mathrm{Tr}_L(m,\Phi) = \pm \sum_{\text{edge-n-colorings } c} \prod_{e=bw} \prod_{i\in c(e)} (\phi_{bw})_{ii}.$$

The sign is $+$ for n odd, as well as for n even when $L = L_+$ is a positive cilia data.

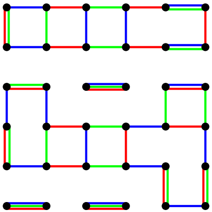
In other words, for planar graphs and diagonal connections, the signs in the formula for the trace are "coherent". The proof uses a result of Thurston [Thu90] that any two dimer covers of a planar bipartite graph are connected by a sequence of face moves.



Sketch of proof: edge-n-colorings

Corollary

For planar bipartite graphs equipped with the identity connection $\Phi = Id$, the trace of a multiweb m is $\pm$ the number of edge-n-colorings of m, with signs as in the above proposition.



Example

The number of edge-3-colorings of the above multiweb–one of which is shown–is $\mathrm{Tr}\,(m, I) = 48 * 24 * 1 * 1 * 1 = 1152$.

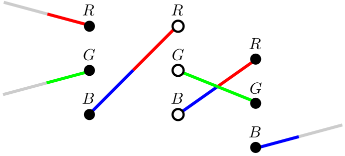
Sketch of proof: finishing

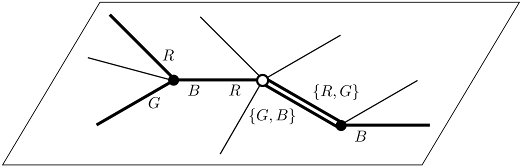
We now turn to the proof of the main formula:

$$\pm \det \tilde{K} = \sum_{m \in \Omega_n} \mathrm{Tr}\,_L(m)$$

Sketch of proof: finishing

$$\pm \det \tilde{K} = \sum_{m \in \Omega_n} \mathrm{Tr}_L(m)$$

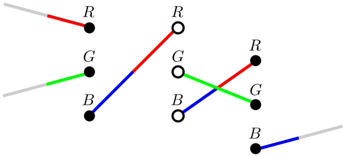


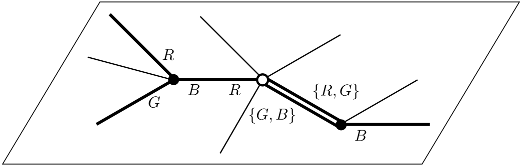


Let $\mathcal{G}_n$ be the **blow-up graph** projecting to $\mathcal{G}$, replacing each vertex of $\mathcal{G}$ with n vertices, and each edge with the complete bipartite graph $B_{n,n}$–each Kasteleyn sign ε_{wb} on an edge "below" is lifted to signs $\varepsilon_{\tilde{w}\tilde{b}}$ on all of the corresponding edges "above".

Sketch of proof: finishing

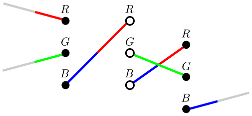
$$\pm \det \tilde{K} = \sum_{m \in \Omega_n} \mathrm{Tr}\,_L(m)$$

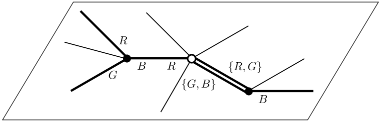




The terms in the determinant $\det \tilde{K}$ are in one-to-one correspondence with dimer covers σ of $\mathcal{G}_n$. Such a dimer cover σ projects to a colored multiweb (m, c) of $\mathcal{G}$, denoted $\sigma \in (m, c)$.

Sketch of proof: finishing





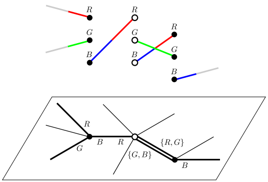
After some manipulation, we can write[17]

$$\det \tilde{K} = \sum_{m \in \Omega_n} \sum_{c} \left(\prod_{e=\tilde{b}\tilde{w}} \varepsilon_{\tilde{w}\tilde{b}} \right) \sum_{\sigma \in (m,c)} (-1)^{\sigma} \prod_{e=\tilde{b}\tilde{w}} (\phi_{bw})_{\tilde{w}\tilde{b}}$$

$$= \sum_{m \in \Omega_n} \left(\sum_{c} \left(\prod \varepsilon_{\tilde{w}\tilde{b}} \right) (-1)^{\sigma_0} \prod_{e=bw} \det (\phi_{bw})_{S_e, T_e} \right).$$

[17] Here σ_0 is the permutation, depending on c but not the $\sigma \in c$, matching the pair S_e, T_e of color subsets in the natural order for each edge e.

Sketch of proof: finishing



$$\det \tilde{K} = \sum_{m \in \Omega_n} \left(\sum_c \left(\prod \varepsilon_{\tilde{w}\tilde{b}} \right) (-1)^{\sigma_0} \prod_{e=bw} \det \left(\phi_{bw}\right)_{S_e, T_e} \right).$$

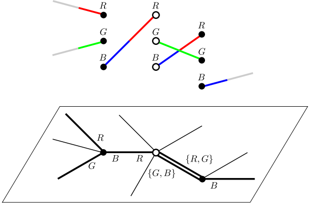
[17]On the other hand, by the determinantal formula for web traces,

$$\mathrm{Tr}_L(m) = \sum_c \prod_v c_v \prod_{e=bw} \det \left(\phi_{bw}\right)_{S_e, T_e}.$$

So it remains to show the signs $\left(\prod \varepsilon_{\tilde{w}\tilde{b}}\right)(-1)^{\sigma_0}$ and $\prod_v c_v$ agree up to a global sign, independent of m.

[17]Here σ_0 is the permutation, depending on c but not the $\sigma \in c$, matching the pair S_e, T_e of color subsets in the natural order for each edge e.

Sketch of proof: finishing



[17]If c is an edge-n-coloring, then there is a **pure dimer cover** σ of $\mathcal{G}_n$–one which matches like colors–projecting to c, in which case the sign coherence is by the formula for planar web traces for diagonal connections, combined with Kasteleyn's theorem: for planar graphs with dimer cover σ_0, the sign $(\prod_{e=bw} \varepsilon_{wb})(-1)^{\sigma_0}$ is a constant, independent of σ_0. The sign coherence for other, non-pure, colorings c follows by comparing to edge-n-colorings. This completes the proof.

[17]Here σ_0 is the permutation, depending on c but not the $\sigma \in c$, matching the pair S_e, T_e of color subsets in the natural order for each edge e.

Application

We can apply our main result to compute web traces. Given a multiweb m, put weights x_e on the edges of m. Then the web trace of m can be extracted from the determinant $\det \tilde{K}$ as the coefficient of the monomial $\prod_{e \in m} x_e^{m_e}$, that is,[18]

$$\mathrm{Tr}_L(m) = \pm \left[\prod_{e \in m} x_e^{m_e} \right] \det \tilde{K}.$$

This is an exponential algorithm, but is useful to compute small examples.

[18]Note web traces are multilinear in the edges.

Table of Contents

Setting: planar surfaces

In this last section, we will be thinking about graphs on a **planar surface**, namely a sphere minus finitely many disjoint closed disks.[19] Let $\mathcal{G}$ be a bipartite graph embedded on a planar surface Σ. We say $\mathcal{G}$ **fills** Σ if every complementary component of $\mathcal{G}$ is a topological disk or disk with a single hole. When $\mathcal{G}$ fills Σ we call the complementary components **faces of** $\mathcal{G}$.

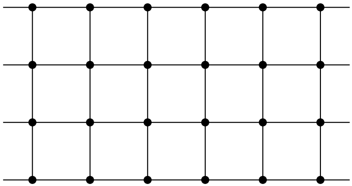


Figure: Graph filling the annulus

[19] Much of what we say is valid for other surfaces too.

Setting: planar surfaces

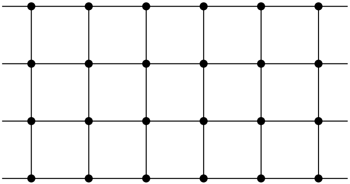


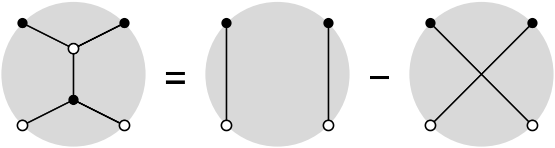
Figure: Graph filling the annulus

A connection Φ on $\mathcal{G}$ is **flat** if its monodromy is trivial around every contractible loop. Throughout this section, we will work exclusively with flat $\mathrm{SL}(3)$ connections.[20]

[20] There is a similar story for $\mathrm{SL}(2)$, but the $\mathrm{SL}(n)$ case is more difficult.

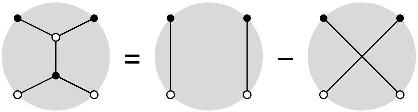
Skein relations for $\mathrm{SL}(3)$ traces

In the introduction, we discussed so-called "skein relations" among $\mathrm{SL}(n)$ trace functions. Especially for $\mathrm{SL}(2)$ and $\mathrm{SL}(3)$ these relations are very helpful in simplifying computations. Here is the **basic** $\mathrm{SL}(3)$ **skein relation**, due to Kuperberg [Kup96].[21]

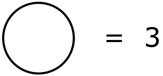


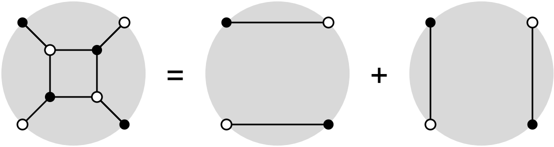
[21] This relation is only equivalent to Kuperberg's relation up to multiplying the coefficients by ± 1.

Skein relations for $\mathrm{SL}(3)$ traces



Here are three more skein relations. The first is **trivial**, and the latter two–the **bigon and square removals**–are consequences of the basic and trivial relations.





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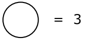
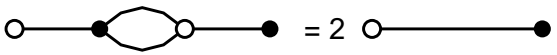
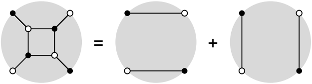


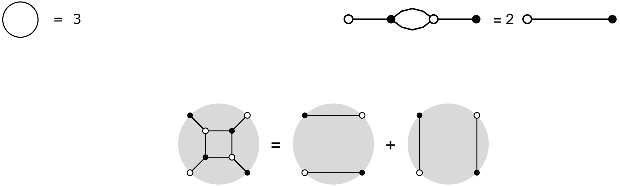




Using these latter three skein relations, one can "reduce" every 3-multiweb $m \in \Omega_3$ to a formal linear combination of **reduced multiwebs**, namely webs whose contractible faces all have at least six sides.[22] This reduction is not unique at the level of multiwebs, but it turns out that it is unique if you consider the webs up to homotopy [Kup96, SW07].

[22] Note the bipartite condition implies all faces have even length.

Skein relations for SL(3) traces



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[22] Note the bipartite condition implies all faces have even length.

Skein relations for $\mathrm{SL}(3)$ traces

We can then rewrite our main result from the previous section as

$$\pm \det \tilde{K}(\Phi) = \sum_{\lambda \in \Lambda_3} C_\lambda \mathrm{Tr}\left(\lambda, \Phi\right)$$

where the coefficients C_λ are natural numbers.

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Theorem (Kuperberg, 1996; Sikora–Westbury 2007)

The trace functions Tr_λ *varying over* $\lambda \in \Lambda_3$ *form a linear basis for the algebra of invariant polynomial functions on the space of flat* $\mathrm{SL}(3)$ *connections on* $\mathcal{G}$ *up to gauge equivalence, i.e. for the algebra of regular functions on the* $\mathrm{SL}(3)$ *character variety.*

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The proof is an application of the so-called "diamond lemma" from algebra. Therefore, at least in principle, the coefficients C_λ can be extracted from the above expression for the determinant of the Kasteleyn matrix. The main problem then is to find ways to extract these coefficients in practice.

Skein relations for $\mathrm{SL}(3)$ traces

Summarizing, in the $\mathrm{SL}(3)$ setting our main result can be rewritten:

Theorem

For a bipartite graph $\mathcal{G}$ in a planar surface, namely a sphere minus $k \geq 0$ disjoint closed disks, equipped with a flat $\mathrm{SL}(3)$ connection Φ,

$$\pm \det \tilde{K}(\Phi) = \sum_{\lambda \in \Lambda_3} C_\lambda \mathrm{Tr}\,(\lambda, \Phi).$$

Probability measures

$$\pm \det \tilde{K}(\Phi) = \sum_{\lambda \in \Lambda_3} C_\lambda \mathrm{Tr}\,(\lambda, \Phi)$$

We now turn to studying probability measures on spaces of webs.

23

24

Probability measures

$$\pm \det \tilde{K}(\Phi) = \sum_{\lambda \in \Lambda_3} C_\lambda \mathrm{Tr}\,(\lambda, \Phi)$$

We now turn to studying probability measures on spaces of webs. As we saw earlier for $n = 1$ and $n = 2$, there is a probability measure on the set of multiwebs Ω_3 with partition function[23][24]

$$Z_{3d} = (Z_d)^3 = \sum_{m \in \Omega_3} \mathrm{Tr}\,(m, I).$$

Here, recall that $\mathrm{Tr}\,(m, I)$ is the number of edge-n-colorings of the multiweb m.

[23] The partition function Z_{nd} is defined for all n in the same way.

[24] By the main result, $|\det \tilde{K}(I)| = Z_{nd}$ for the identity connection.

Probability measures

$$\pm \det \tilde{K}(\Phi) = \sum_{\lambda \in \Lambda_3} C_\lambda \mathrm{Tr}\,(\lambda, \Phi)$$

$$Z_{3d} = \sum_{m \in \Omega_3} \mathrm{Tr}\,(m, I).$$

Moreover, the above theorem allows us to define a natural probability measure on the set Λ_3 of equivalence classes of reduced webs–but *not* on the subset $\mathrm{Red}(\Omega_3) \subset \Omega_3$ of reduced multiwebs, i.e. without equivalence. Specifically, the probability measure on Λ_3 is defined by

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$$\mathrm{Pr}(\lambda) = \frac{C_\lambda \mathrm{Tr}\,(\lambda, I)}{Z_{3d}}.$$

Probability measures

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$$Z_{3d} = \sum_{m \in \Omega_3} \mathrm{Tr}\,(m, I) \qquad \mathrm{Pr}(\lambda) = \frac{C_\lambda \mathrm{Tr}\,(\lambda, I)}{Z_{3d}}.$$

Open problems

- Find a canonical probability measure on the subset $\mathrm{Red}(\Omega_3) \subset \Omega_3$ of reduced multiwebs, without equivalence.[a]
- Find a notion of "reduced multiweb" $m \in \mathrm{Red}(\Omega_n) \subset \Omega_n$.[b] Then repeat the previous problem for $n = n$.

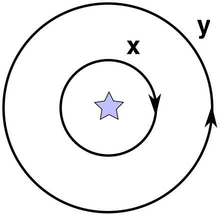
[a] This has a simple solution for $n = 2$.

[b] As we saw earlier, at least for the annulus this is known. But even for the pair of pants it is tough. This problem is also relevant to quantum topology.

Case of the annulus: reduced webs

Proposition

For the sphere with 2 holes, i.e. the annulus, Λ_3 consists of homotopy classes of disjoint oriented noncontractible cycles.



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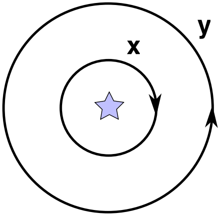
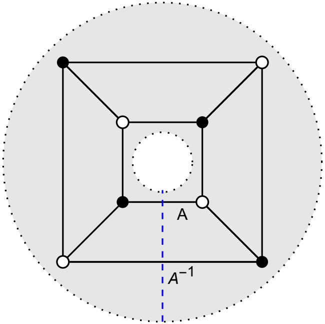


The proof is by Euler characteristic considerations. Moreover, by Sikora–Westbury's theorem, the traces for these multicurves are a linear basis for the algebra of regular functions on the $\mathrm{SL}(3)$ character variety of the annulus.

Case of the annulus: reduced webs

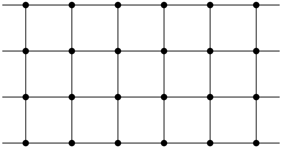


$$\pm \det \tilde{K}(\text{annulus}) = \sum_{\lambda \in \Lambda_3} C_\lambda \mathrm{Tr}\,(\lambda) = \sum_{j,k \geq 0} C_{j,k} \mathrm{Tr}\,(A)^j \mathrm{Tr}\,(A^{-1})^k$$

Here, A is the monodromy around one of the cycle generators.

Case of the annulus: model graph

Consider now the $2m$ by n square grid $\mathcal{G}_{2m,n}$ on the annulus.[25]



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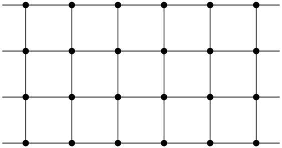

Since the connection is diagonalizable, we can write

$$\det \tilde{K} = \det K(x) \det K(y) \det K(z)$$

for $x, y, z = 1/(xy)$ the eigenvalues of A, and $K(x)$ the one-variable Kasteleyn matrix.

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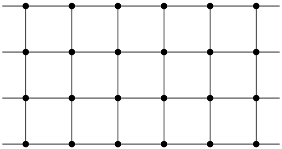

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for $x, y, z = 1/(xy)$ the eigenvalues of A, and $K(x)$ the one-variable Kasteleyn matrix. By thinking about the eigenvectors and eigenvalues of the (large) one-variable Kasteleyn matrix–$f_{j,k}(x,y) = e^{2\pi ijx/(2m)} \sin \frac{\pi ky}{n+1}$ and $\lambda_{j,k} = \zeta e^{2\pi ij/(2m)} + \zeta^{-1} e^{-2\pi ij/(2m)} + 2i \cos \frac{\pi k}{n+1}$–we compute

[25] We require m is odd and n is even.

Case of the annulus: model graph

$$\det \tilde{K} = \pm \prod_{j=1}^{n/2} \left(1 + 3v\alpha_j^{2m} + 3u\alpha_j^{4m} + \alpha_j^{6m}\right) \left(1 + 3v\alpha_j^{-2m} + 3u\alpha_j^{-4m} + \alpha_j^{-6m}\right) \quad (1)$$

for $u = \frac{1}{3}\mathrm{Tr}\,(A)$, $v = \frac{1}{3}\mathrm{Tr}\,(A^{-1})$, and $\alpha_j = -\cos\theta_j + \sqrt{1 + \cos^2\theta_j}$ for $\theta_j = \frac{\pi j}{n+1}$.

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$$P(u,v) := \frac{\det \tilde{K}(u,v)}{\det \tilde{K}(1,1)} = \sum_{p,q \geq 0} c_{p,q} u^p v^q$$

where $c_{p,q}$ is the probability–in the sense above–of a reduced web of type p, q.

Case of the annulus: model graph

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where $c_{p,q}$ is the probability–in the sense above–of a reduced web of type p, q. By interpreting $P(u,v)$ as the probability generating function for an n-step random walk in $\mathbb{Z}^2$ starting from $(0,0)$ we compute the mean values

Case of the annulus: model graph

$$\bar{X} = \bar{Y} = \sum_{j=1}^{n/2} \frac{3\alpha_j^{2m}}{(1+\alpha_j^{2m})^2}$$

of the number of nontrivial loops of each orientation on the annulus.

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$$P(u,v) = \frac{\det \tilde{K}(u,v)}{\det \tilde{K}(1,1)} = 1 + o_q(1)$$

for $q = e^{-\pi\tau}$. So the probability of having either type of crossing tends to 0 as $q \to 0$ as expected.

Case of the annulus: model graph

$$\bar{X} = \bar{Y} = \sum_{j=1}^{n/2} \frac{3\alpha_j^{2m}}{(1+\alpha_j^{2m})^2}$$

of the number of nontrivial loops of each orientation on the annulus. In the limit $m, n \to \infty$, $m/n \to \tau >> 1$ of a long thin annulus, we compute

$$P(u,v) = \frac{\det \tilde{K}(u,v)}{\det \tilde{K}(1,1)} = 1 + o_q(1)$$

for $q = e^{-\pi\tau}$. So the probability of having either type of crossing tends to 0 as $q \to 0$ as expected. Moreover, the probability $c_{j,k}$ of a (j,k)-crossing is to leading order $q^{\delta_{j,k}} 3^{j+k}$ where the **crossing exponent** is $\delta_{j,k} = \lceil \frac{2(j^2+jk+k^2)}{3} \rceil$.

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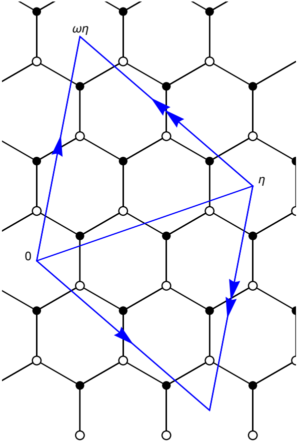
Pair of pants

The pair of pants, i.e. sphere minus 3 disks, is more complicated. Let $\omega = e^{\pi i/3}$ and let $\eta \in \mathbb{C}$ have the form $\eta = a + b\omega$ where $a, b \in \mathbb{Z}_{\geq 0}$. Each such **Eisenstein integer** η indexes a reduced web W_η as in the following picture



Pair of pants

Proposition

A reduced web on a pair of pants is a disjoint union of a collection of loops and at most one of the webs W_η.

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So letting $A, B, C = AB$ be the monodromies around the punctures of a flat connection on the pair of pants, we have

$$\det \tilde{K} = \sum_{i_1,i_2,j_1,j_2,k_1,k_2,\eta} C_{\vec{i}} \mathrm{Tr}(A)^{i_1} \mathrm{Tr}(A^{-1})^{i_2} \mathrm{Tr}(B)^{j_1} \mathrm{Tr}(B^{-1})^{j_2} \mathrm{Tr}(C)^{k_1} \mathrm{Tr}(C^{-1})^{k_2} \mathrm{Tr}(W_\eta)$$

for some natural numbers $C_{\vec{i}} = C_{i_1,i_2,j_1,j_2,k_1,k_2,\eta}$.

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In our paper, we do this under the very simplified assumption that two of the boundary components are in adjacent faces of $\mathcal{G}$.

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Thanks!

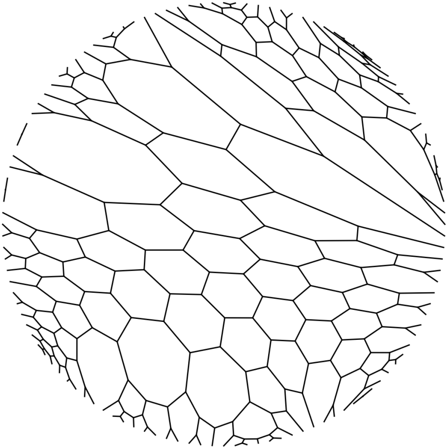

Figure: [26]Uniform random reduced 3-web for a disk with 3n black boundary vertices, sampled here for $n = 40$.

[26]Many pretty pictures (including this one) were courtesy of Rick Kenyon.