Quantum Field Theory
for
Mathematicians

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CUNY, August 2023

Outline

QFT is a vast subject, with numerous deep connections to various aspects of mathematics.

In these lectures I'll try to connect the topic of this workshop to one aspect of QFT, the theory of scattering amplitudes.

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1. Introduction to Amplitudes
2. "Trees" $\rightsquigarrow$ plabic graphs & the amplituhedron
3. "Loops" $\rightsquigarrow$ cluster algebras & tropical fans

1. Introduction to Amplitudes

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- a choice of momentum $\vec{p} \in \mathbb{R}^{D-1}$ and
- a choice of unitary irreducible representation R of the stabilizer of the energy-momentum vector $p = \left(\sqrt{m^2 + |\vec{p}|^2}, \vec{p}\right)$ in $\mathrm{Spin}^+(1, D-1)$.

The stabilizer is $\begin{cases} SO(D-2) & \text{if } m = 0 \\ SO(D-1) & \text{if } m > 0 \end{cases}$

"Physics-Talk" for Mathematicians

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Special names indicate representations:

scalar	$[0, \ldots, 0]$
vector, photon, gauge boson	$[1, 0, \ldots, 0]$
graviton	$[2, 0, \ldots, 0]$

"Physics-Talk" for Mathematicians

A particle is called a boson (fermion) if $\mathcal{R}$ is a tensorial (spinorial) representation of $Spin^{+}(1, D-1)$.

[$Spin^{+}(1, D-1)$ is a double cover of $SO^{+}(1, D-1)$.

A tensorial rep. is a "true" rep. of $SO^{+}(1, D-1)$.

A spinorial rep. is one that is "only" a projective rep., not a "true" rep., of $SO^{+}(1, D-1)$]

Fock Space & the S-Matrix

One way to describe a QFT is to provide a list of particle types $(m_1, R_1), (m_2, R_2), \ldots$ (not necessarily finite, not necessarily distinct)

Fock Space & The S-Matrix

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The one-particle Hilbert space is

$$H = \bigoplus_{\text{types } i=1,2,\ldots} \; \bigoplus_{\vec{p} \in \mathbb{R}^{D-1}} \mathbb{C}^{\dim(R_i)}$$

Fock Space & the S-Matrix

$$H = \bigoplus_{\text{types } i=1,2,\dots} \ \bigoplus_{\vec{p} \in \mathbb{R}^{D-1}} \mathbb{C}^{\dim(R_i)}$$

and the Fock space is the subspace of

$$F = \bigoplus_{n=0}^{\infty} H^{\otimes n}$$

obtained by symmetrizing (antisymmetrizing) over identical bosons (fermions).

Note that there is a naturally induced action $L: F \to F$ of $SO^+(1, D-1)$ on F.

Fock Space & The S-Matrix

An S-matrix is a unitary transformation

$$S : \mathcal{F} \longrightarrow \mathcal{F}$$

that transforms appropriately under $SO^+(1, D-1)$ (Lorentz) transformations acting simultaneosly on all particles: $L^\dagger S L = S \quad \forall L.$

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that transforms appropriately under $SO^+(1, D-1)$ (Lorentz) transformations acting simultaneosly on all particles: $L^\dagger S L = S \ \ \forall L$.

Physically, S represents the evolution of some collection of "incoming" particles to some collection of "outgoing" particles.

Fock Space & the S-Matrix

Unitarity: If we write $S = 1 + iT$ then

$$S^\dagger S = 1 \implies -i(T - T^\dagger) = T^\dagger T$$

This will be the basis of "factorization".

Energy-momentum conservation:

$$T(p_1, p_2, \cdots \rightarrow \cdots, p_n)$$

vanishes unless $p_1 + p_2 + \cdots = \cdots + p_n$

Path Integrals and Perturbation Theory

It is a "folk theorem" in physics that if $S(g)$ is a one-parameter family of S-matrices with $S(0) = 1$, then there exists a path integral which computes $S(g)$, at least asymptotically near $g = 0$.

Path Integrals and Perturbation Theory

Consider the "$D=0$" path integral

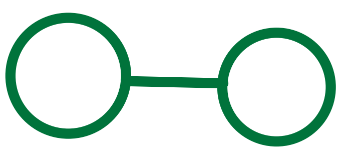
$$\frac{\int_{-\infty}^{\infty} d\phi \, \exp\left(-\frac{1}{2} m^2 \phi^2 - \frac{g}{6} \phi^3\right)}{\int_{-\infty}^{\infty} d\phi \, \exp\left(-\frac{1}{2} m^2 \phi^2\right)}$$

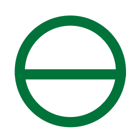
$$= 1 + \frac{5}{24} \frac{g^2}{m^6} + \frac{385}{1152} \frac{g^4}{m^{12}} + \mathcal{O}(g^6)$$

Path Integrals and Perturbation Theory

$$1 + \frac{5}{24}\frac{g^2}{m^6} + \frac{385}{1152}\frac{g^4}{m^{12}} + O(g^6)$$

There are simple graphical rules for computing these terms by drawing all "Feynman diagrams" weighing each edge with a factor of $1/m^2$ and each vertex with a factor of g, divided by symmetry factors to avoid overcounting:

 $\frac{1}{2^3}\frac{g^2}{m^6}$

 $\frac{1}{2\cdot 3!}\frac{g^2}{m^6}$

Path Integrals and Perturbation Theory

When generalized to $D>0$ we get functional integrals of the form

Laplacian on $\mathbb{R}^D$

$$\int D\phi(x)\, \exp\left[-\int d^D x\, \phi(-\partial^2+m^2)\phi \;+ O(g)\right]$$

Several comments follow:

Path Integrals and Perturbation Theory

When generalized to $D>0$ we get functional integrals of the form

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$$\int \mathcal{D}\phi(x)\, \exp\left[-\int d^D x\, \phi(-\partial^2+m^2)\phi \quad + O(g)\right]$$

Several comments follow:

This path integral describes a <u>single</u> type of particle, of mass m. To get more types we use more fields ϕ_1, ϕ_2, etc.

Path Integrals and Perturbation Theory

Laplacian on $\mathbb{R}^D$

$$\int D\phi(x) \exp\left[-\int d^D x \frac{1}{2} \phi(-\partial^2 + m^2)\phi \quad + O(g) \right]$$

Edges in feynman diagrams are now weighted not by $1/m^2$ but by the functional inverse

$$(-\partial^2 + m^2)^{-1} = \frac{1}{p^2 + m^2}$$ (in Fourier representation)

Note: in $\mathbb{R}^{1,D-1}$ $\quad p^2 = -E^2 + |\vec{p}|^2$

Path Integrals and Perturbation Theory

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$$\int D\phi(x) \exp\left[-\int d^D x \frac{1}{2}\phi(-\partial^2 + m^2)\phi + O(g)\right]$$

Edges in Feynman diagrams are now weighted not by $1/m^2$ but by the functional inverse

$$(\partial^2 + m^2)^{-1} = \frac{1}{p^2 + m^2}$$ (in Fourier representation)

We integrate over all field configurations, not just those on-shell where $p^2 = -m^2$.

Path Integrals and Perturbation Theory

I will not explain in detail how to compute the S-matrix from such a path integral; the rule for computing

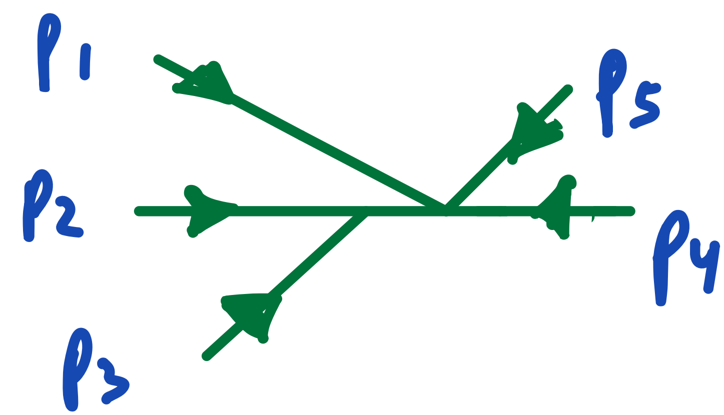
$$T(p_1, p_2, \cdots \rightarrow \cdots, p_n) \text{ involves}$$

$$\int \mathcal{D}\tilde{\phi}(p)\, \tilde{\phi}(p_1)\tilde{\phi}(p_2)\cdots\tilde{\phi}(-p_n)\, \exp\Big[-\tfrac{1}{2}\int d^D p\; \tilde{\phi}(p)(p^2+m^2)\tilde{\phi}(-p) + \mathcal{O}(g)\Big]$$

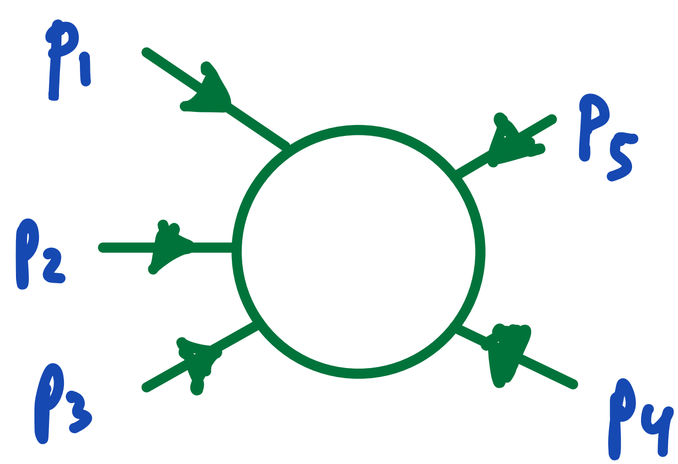
The Feynman diagram rule for the extra insertions is to have n external edges:

Path Integrals and Perturbation Theory

e.g.



or



Henceforth we adopt the convention that outgoing particles have negative energy, so we draw all arrows incoming. The diagrams evaluate to 0 unless $p_1 + p_2 + p_3 + p_4 + p_5 = 0$.

Path Integrals and Perturbation Theory

This is a good time to mention the definition

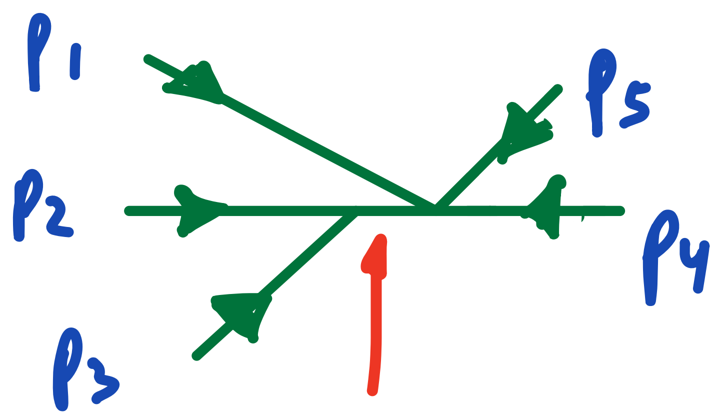
$$T(p_i) = (2\pi)^D \delta^D(p_1 + p_2 + \cdots + p_n)\, A(p_i)$$

of the scattering amplitude $A(p_i)$.

T is a distribution, A is a function (with singularities).

Path Integrals and Perturbation Theory

Energy-momentum is conserved at each vertex; for tree diagrams the energy-momentum carried by each leg is a sum of a subset of those of the external legs:



For example, here $\longrightarrow$ carries energy-momentum $p_2 + p_3 = -(p_1 + p_4 + p_5)$

Path Integrals and Perturbation Theory

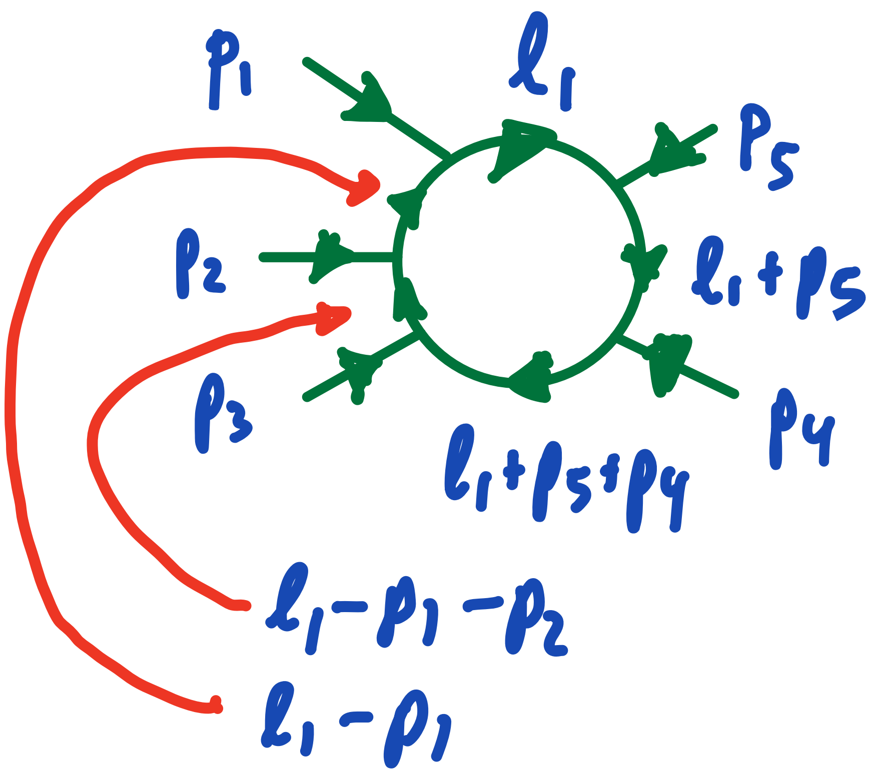
The "$+O(g)$" terms in the path integral determine the factors associated to <u>vertices</u> in Feynman diagrams — just like in our $D=0$ example.

If we assume such terms are polynomial in $\phi(x)$ and its derivatives, then

tree amplitudes evaluate to rational functions of the energy-momentum vectors p_i, with poles <u>only</u> where some subset "goes on-shell".

Path Integrals and Perturbation Theory

However, loop diagrams (with one or more closed loops) have one or more independent loop momenta that must be integrated over

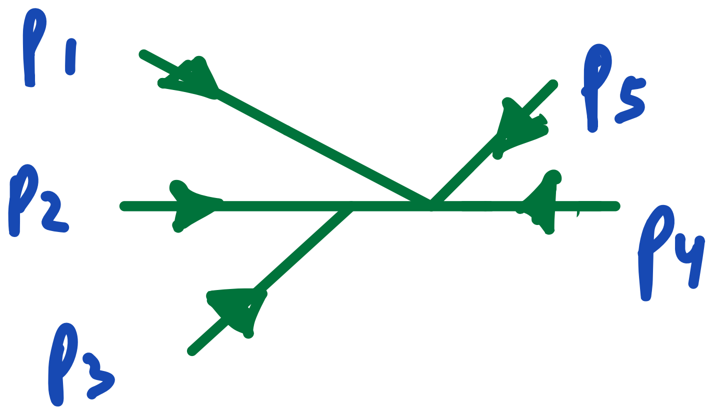


$$= \int d^D \ell_1 \left(\begin{array}{c} \text{some rational} \\ \text{function of} \\ \ell_1 \text{ and the } p_i \end{array} \right)$$

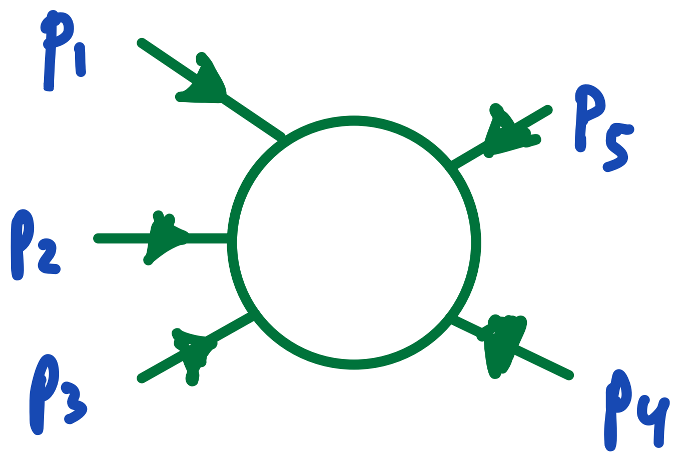
much more complicated!

Path Integrals and Perturbation Theory

e.g.



or



Loop diagrams always have more vertices than tree diagrams — so they are sub-dominant in the small g asymptotic expansion.

Tree-level Unitarity

Elements of the scattering matrix $S = 1 + iT$ should be understood as distributions.

The physical principle of causality (cause should preceed effect!) dictates that the proper way to understand

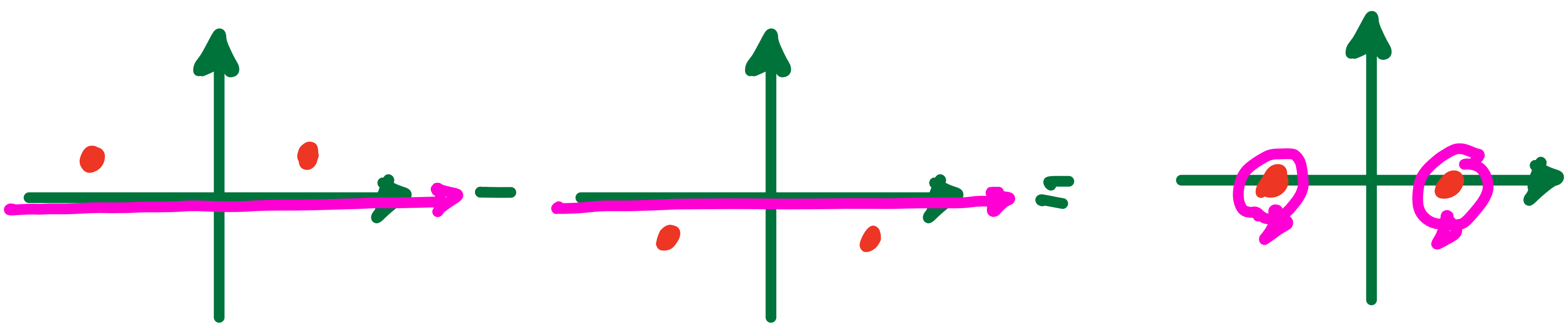
$$\frac{1}{p^2+m^2} \quad \text{is} \quad \lim_{\varepsilon \to 0} \frac{1}{p^2+m^2-i\varepsilon}$$

Tree-level Unitarity

Then using the distributional identity

$$\lim_{\varepsilon \to 0} \left(\frac{1}{p^2 + m^2 + i\varepsilon} - \frac{1}{p^2 + m^2 - i\varepsilon} \right) = 2\pi i\, \delta(p^2 + m^2)$$

[Imagine integrating both sides times a smooth function of p in the complex E plane:

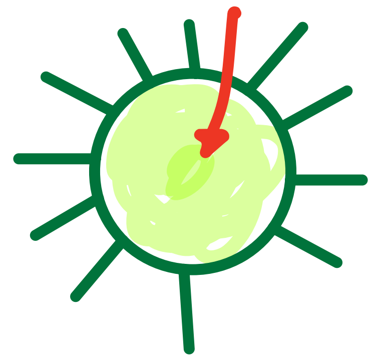


]

Tree-level Unitarity

In pictures:

drawing a blob like this means we sum over all tree graphs

$$T(p_1, \dots, p_n) = $$ 

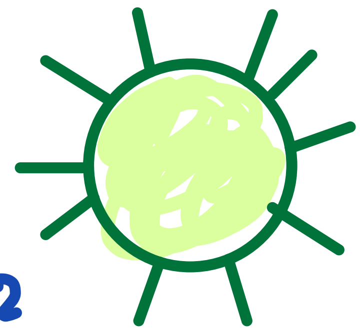
This has poles whenever

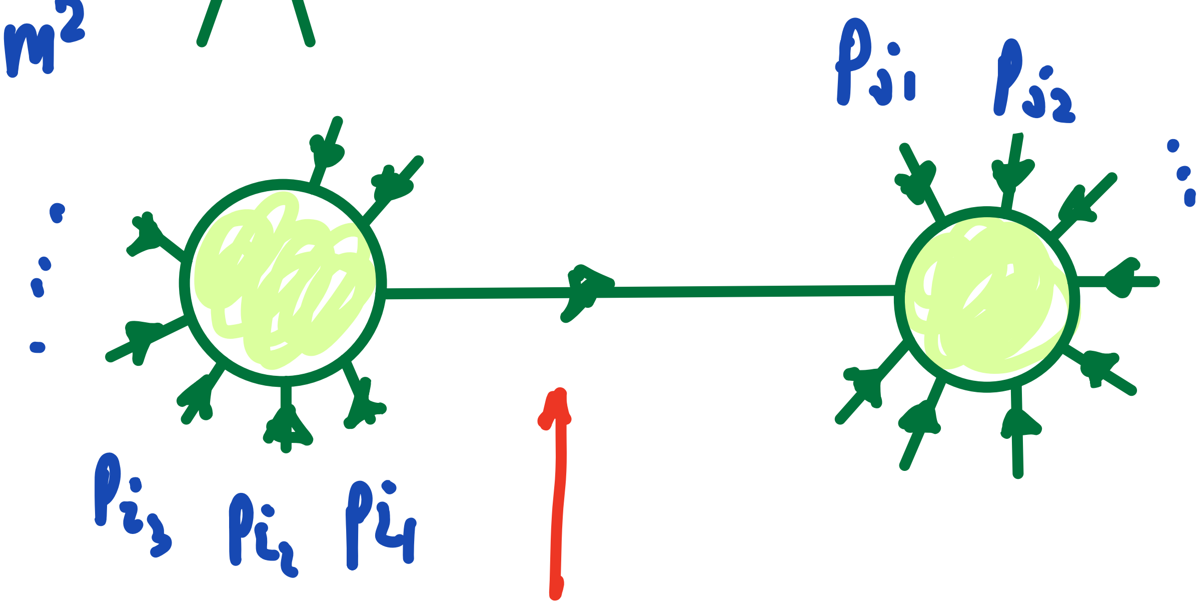
$$(p_{i_1} + p_{i_2} + \cdots)^2 = m^2 = (p_{j_1} + p_{j_2} + \cdots)^2$$

for two complementary subsets of the p_i.

Tree-level Unitarity

Factorization/unitarity $\quad -i(T-T^\dagger)=T^\dagger T$ implies

$\mathrm{Res}_{(p_{i_1}+p_{i_2}+\cdots)^2=m^2}$ 

$=$ 

$$p_{i_1}+p_{i_2}+\cdots = -p_{j_1}-p_{j_2}-\cdots$$

Tree-level Unitarity

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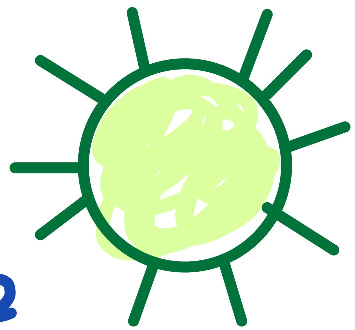

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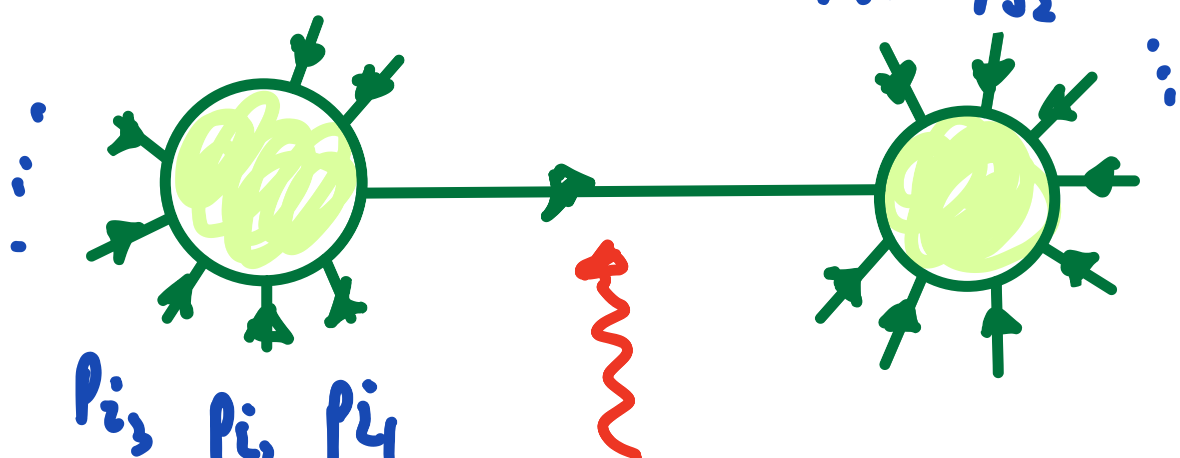

If T describes more than one particle type we must sum over all types exchanged.

Tree-level Unitarity

The main takeaway for mathematicians is that at tree-level, we are in a world populated by infinite collections of intricate rational functions of "kinematic data" (energies momenta of particles) satisfying very stringent constraints on their singularities.

Part 2: Trees, Plabic Graphs, and the Amplituhedron

Some References Part 2.

Modern Amplitudes Methods

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- H. Elvang, Y-t Huang 1308.1697
- C. Cheung 1708.03872

Positive Grassmannian, plabic graphs, on-shell diagrams

- A. Postnikov math/0609764
- Arkani-Hamed et al 1212.5605

The Amplituhedron

- N. Arkani-Hamed, J. Trnka 1312.2007

Positive Geometry and Canonical Forms

- N. Arkani-Hamed, Y. Bai, T. Lam 1703.04541

Massless Particles in $D=4$

Now I will specialize to a QFT describing a collection of massless vector bosons in $D=4$, which has particularly rich structure.

Massless Particles in $D=4$

Now I will specialize to a QFT describing a collection of massless vector bosons in $D=4$, which has particularly rich structure.

Massless particles in $D=4$ are special because the stabilizer $SO(2)$ is abelian

- all irreducible representations are 1-dimensional
- and has an infinite cover, not a double cover as mentioned before.

Helicity in $D=4$

Quantum mechanics allows only projective representations of $SO(2)$ of the form

$$\exp(i\theta) \longrightarrow \exp(ih\theta)$$

for integer or half-integer values of

h = <u>helicity</u>

"Gauge bosons" have $h = \pm 1$.

Spinor Helicity

The $SO(2)$ transformations of a particle state are nicely exhibited using spinor helicity variables

$$p = (E, p_1, p_2, p_3) \text{ with } p^2 = -E^2 + p_1^2 + p_2^2 + p_3^2 = 0$$

$$\Rightarrow \det \begin{pmatrix} E + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & E - p_3 \end{pmatrix} = 0$$

Spinor Helicity

$$\Rightarrow \det\begin{pmatrix} E+p_3 & p_1-ip_2 \\ p_1+ip_2 & E-p_3 \end{pmatrix} = 0$$

This matrix has rank 1 so it can be represented as an outer product

$$\lambda\tilde{\lambda} = \begin{pmatrix} \lambda_1\tilde{\lambda}_1 & \lambda_1\tilde{\lambda}_2 \\ \lambda_2\tilde{\lambda}_1 & \lambda_2\tilde{\lambda}_2 \end{pmatrix} \text{ of } \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \; \tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_1 \\ \hat{\lambda}_2 \end{pmatrix}$$

Spinor Helicity

This description is redundant since

$\lambda \to t\lambda$, $\tilde{\lambda} \to t^{-1}\tilde{\lambda}$ leaves p unchanged

for $t \in \mathbb{C}^*$.

Spinor Helicity

This description is redundant since

$\lambda \to t\lambda$, $\tilde{\lambda} \to t^{-1}\tilde{\lambda}$ leaves p unchanged

for $t \in \mathbb{C}^*$.

This is useful since the $so(2)$ stabilizer of p inside $so^+(1,3)$ acts via

$\lambda \to e^{-i\theta}\lambda$, $\tilde{\lambda} \to e^{+i\theta}\tilde{\lambda}$

[$\mathbb{C}^*$ appearing above is the complexification of $so(2)$]

Massless Amplitudes in $D=4$

A scattering amplitude of n massless particles is thought of as a function of n pairs of spinor helicity variables

$$A(\lambda_1, \tilde{\lambda}_1; \lambda_2, \tilde{\lambda}_2; \ldots; \lambda_n, \tilde{\lambda}_n) \qquad p_i = \lambda_i \tilde{\lambda}_i$$

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$$A(\lambda_1, \tilde{\lambda}_1; \lambda_2, \tilde{\lambda}_2; \dots; \lambda_n, \tilde{\lambda}_n) \qquad p_i = \lambda_i \tilde{\lambda}_i$$

and must, as a consequence of $SO^+(1, D-1)$ Lorentz invariance, satisfy

$$A(t_i \lambda_i, t_i^{-1} \tilde{\lambda}_i) = \left(\prod_i t_i^{-2h_i}\right) A(\lambda_i, \tilde{\lambda}_i)$$

Massless Amplitudes in $D=4$

This requirement is strong enough to <u>uniquely</u> determine 3-particle amplitudes.

To that end I'll need the notation

$$\langle ij \rangle = \lambda_{i,1}\lambda_{j,2} - \lambda_{i,2}\lambda_{j,1}$$

$$[ij] = \tilde{\lambda}_{i,1}\tilde{\lambda}_{j,2} - \tilde{\lambda}_{i,2}\tilde{\lambda}_{j,1}$$

Three-Particle Massless Amplitudes

$$A(1,2,3) = g \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3} \times \langle 13\rangle^{h_2-h_1-h_3}$$

if $h_1+h_2+h_3 \leq 0$ or

$$\tilde{g}\,[12]^{h_3-h_1-h_2}\,[23]^{h_1-h_2-h_3} \times [13]^{h_2-h_1-h_3}$$

if $h_1+h_2+h_3 \geq 0$

where g, $\tilde{g}$ are undetermined numbers.

Three-Particle Massless Amplitudes

Note: spacetime reflection symmetry exchanges $\lambda \longleftrightarrow \tilde{\lambda}$, hence

$$\langle\, , \rangle \longleftrightarrow [\, ,]$$

and hence requires $g = \tilde{g}$.

Gauge Theory

The simplest nontrivial S-matrix of massless vector particles $(h = \pm 1)$ has three-particle amplitudes

denotes $h_1 = +1$ etc

$$A(1^+, 2^+, 3^-) = g \frac{[12]^3}{[23][13]}$$

$$A(1^+, 2^-, 3^-) = g \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 13 \rangle}$$

$$A(1^+, 2^+, 3^+) = A(1^-, 2^-, 3^-) = 0 \quad \star$$

Gauge Theory

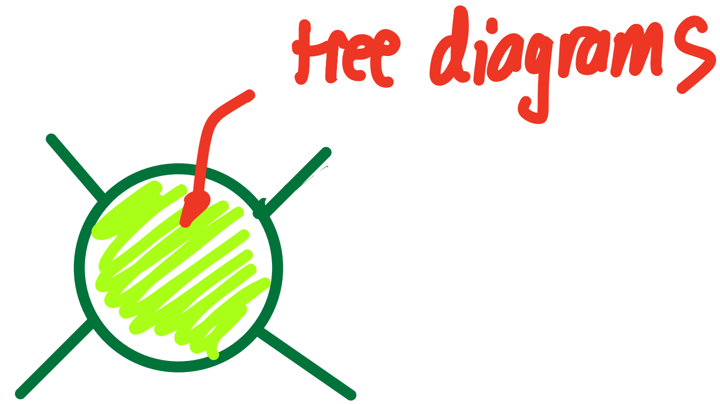
In this (☆) case, all amplitudes with >3 particles can be uniquely determined by factorization/unitarity under one assumption:

that they are rational functions of the spinor helicity variables.

This corresponds to the tree-level (small g) approximation in perturbation theory.

The 4-gauge Boson Amplitude

We learn something very interesting by considering the factorization of

$A(1^-, 2^-, 3^+, 4^+) =$ 

Consideration of $\lambda, \tilde{\lambda}$ scaling tells us that the amplitude is a linear combination of three possible terms.

The 4-gauge Boson Amplitude

There are only 3 rational functions that have the correct $\lambda, \tilde{\lambda}$ scaling, so A must be a linear combination

$$A = c_1 \frac{\langle 12\rangle [34]^2}{[12][23]\langle 23\rangle} +$$

$$c_2 \frac{\langle 12\rangle^2 [34]^2}{[13][23]\langle 13\rangle\langle 23\rangle} +$$

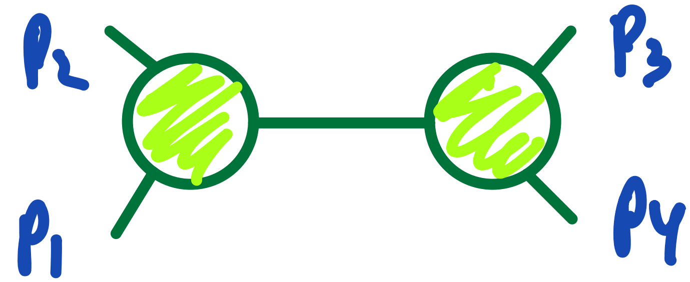
$$c_3 \frac{[34]^2 \langle 12\rangle}{[12][13]\langle 13\rangle}$$

for some c_i to be determined

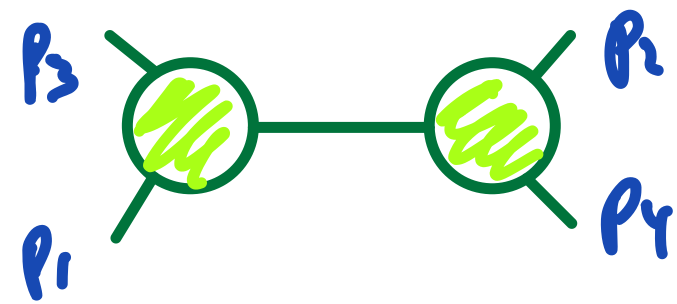
The 4-gauge Boson Amplitude

To determine the c_i we impose unitarity

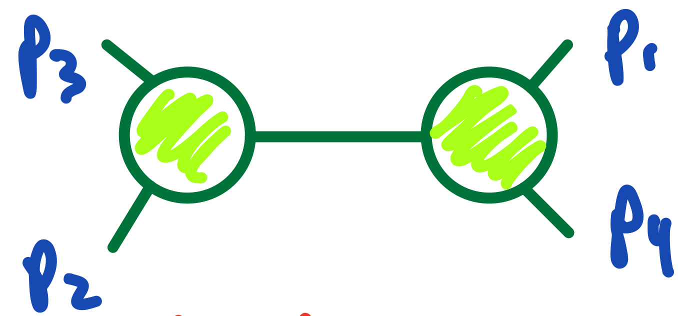
$$\underset{(p_1+p_2)^2=0}{\mathrm{Res}} \; A =$$



$$\underset{(p_1+p_3)^2=0}{\mathrm{Res}} \; A =$$



$$\underset{(p_2+p_3)^2=0}{\mathrm{Res}} \; A =$$



The right-hand sides involve known amplitudes!

The 4-gauge Boson Amplitude

A simple calculation reveals that if there is only a single type of gauge boson particle, then the only solution is $c_i = 0$ — boring!

The 4-gauge Boson Amplitude

Suppose instead there are several types and the three-particle amplitudes are

$$A(1_a^+, 2_b^+, 3_c^-) = g_{abc} \frac{[12]^3}{[23][13]}$$

$$A(1_a^+, 2_b^-, 3_c^-) = g_{abc} \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 13\rangle}$$

Here the subscripts label "particle type";

The 4-gauge Boson Amplitude

Then factorization of the 4-particle amplitude allows a non-zero solution only if

$$\sum_e g_{abe}\, g_{cde} + g_{bce}\, g_{ade} + g_{cae}\, g_{bde} = 0$$

for all a, b, c, d. (The $\sum_e$ arises from summing over all particle types in the exchange between three-particle amplitudes.)

The 4-gauge Boson Amplitude

Evidently, massless particles with $h = \pm 1$ must come in collections associated with some Lie algebra.

The 4-gauge Boson Amplitude

Let T_a be a set of matrices that furnish the adjoint representation of some Lie algebra. Then the solution of factorization is

$$A(1_a^-, 2_b^-, 3_c^+, 4_d^+) =$$

$$\mathrm{Tr}[T_a T_b T_c T_d] \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$+ \mathrm{Tr}[T_a T_b T_d T_c] \frac{\langle 12\rangle^3}{\langle 24\rangle\langle 43\rangle\langle 31\rangle}$$

$$+ \cdots$$

Ordered Partial Amplitudes

More generally, the structure of an n point amplitude is

$$A(1_{a_1}, 2_{a_2}, \ldots, n_{a_n}) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{\sigma(a_1)} \cdots T^{\sigma(a_n)}) \, A(\sigma(1), \sigma(2), \ldots, \sigma(n))$$

where this is the ordered partial amplitude.

Ordered Partial Amplitudes

The ordered partial amplitude may be computed directly by summing all Feynman diagrams that are planar with respect to a fixed cyclic ordering of the n external edges/particles.

These amplitudes only have poles associated to factorizations that preserve the order.

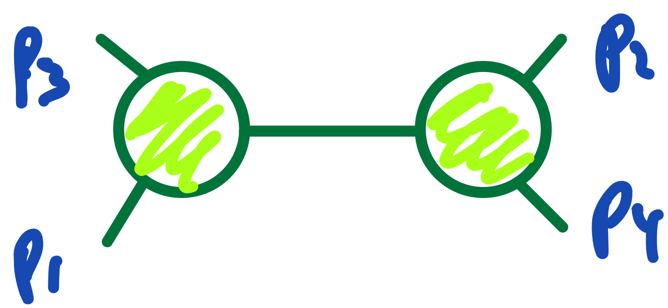
Ordered Partial Amplitudes (henceforth "amplitudes")

e.g. the ordered partial amplitude

$A(1,2,3,4)$ does not have a pole at

$(p_1+p_3)^2=(p_2+p_4)^2=0$, which would

correspond to the out-of-order factorization



MHV Amplitudes

The complexity of gauge boson amplitudes is stratified by the number of negative helicity particles.

$$A(1^+, 2^+, \dots, n^+) = 0$$

$$A(1^+, 2^+, \dots, i^-, \dots, n^+) = 0$$

$$A(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \frac{\langle ij \rangle^4}{\prod_i \langle i, i+1 \rangle}$$

N^kMHV Amplitudes

$$A(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij \rangle^4}{\prod_i \langle i, i+1 \rangle}$$

These are called MHV amplitudes.

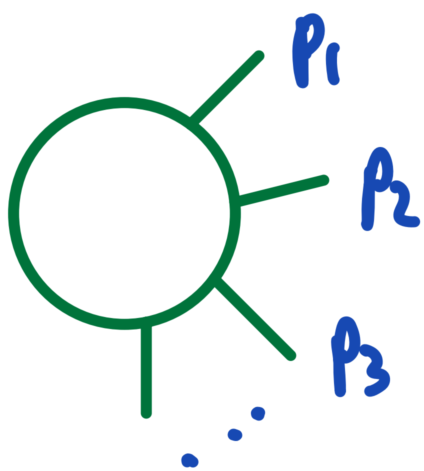
NMHV have three $h=-1$ particles,

N^2MHV have four, etc.

The complexity of amplitudes grows with k (until $k = n/2$ after which pattern flips it back.)

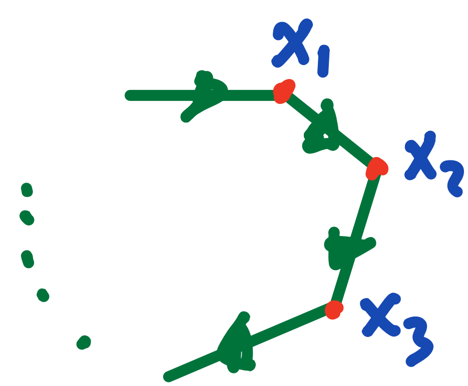
Grassmannians

The planar structure allows for a connection to Grassmannians



Given a collection of ordered $p_1, \ldots, p_n$ satisfying $p_1 + \cdots + p_n = 0$ define $p_i = x_{i+1} - x_i$

These are the vertices of a polygon in $\mathbb{R}^D$ whose edges are the vectors p_i



Grassmannians

Such a configuration is equivalent to a collection of n ordered points $Z_i \in \mathbb{P}^3$.

To see this: denote each homogeneous coordinate

$$Z_i = (\lambda_{i,1}, \lambda_{i,2}, \mu_{i,1}, \mu_{i,2})$$

and define the 2×2 matrix x_i as the solution to the four equations

Grassmannians

$$\mu_i = x_i \lambda_i \qquad \mu_{i-1} = x_i \lambda_{i-1}$$

Note that by construction

$$(x_i - x_{i+1}) \lambda_i = 0$$

so (at least in a patch where λ_i is not identically zero) $x_i - x_{i+1}$ has rank 1, so it can be interpreted as an on-shell energy-momentum vector p_i.

Grassmannians

The takeaway is that the kinematic configuration space is that of n ordered, but otherwise arbitrary, points in $\mathbb{P}^3$.

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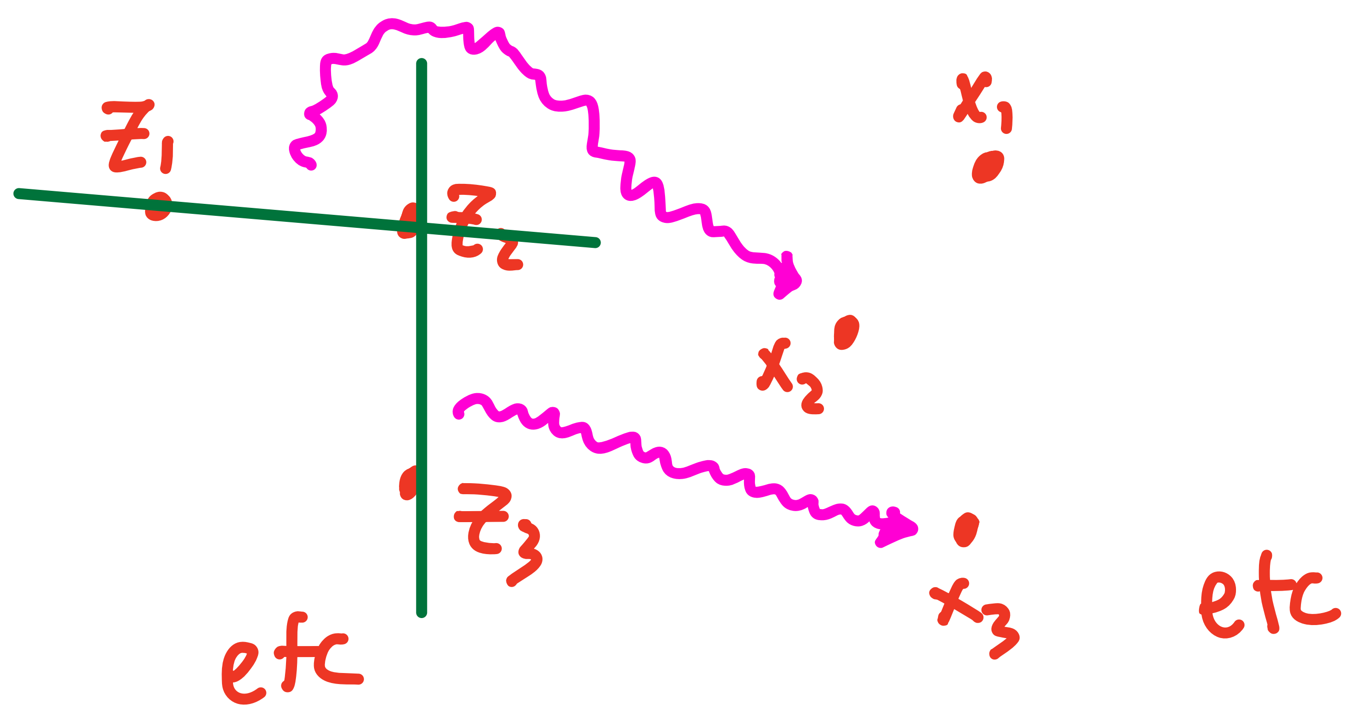
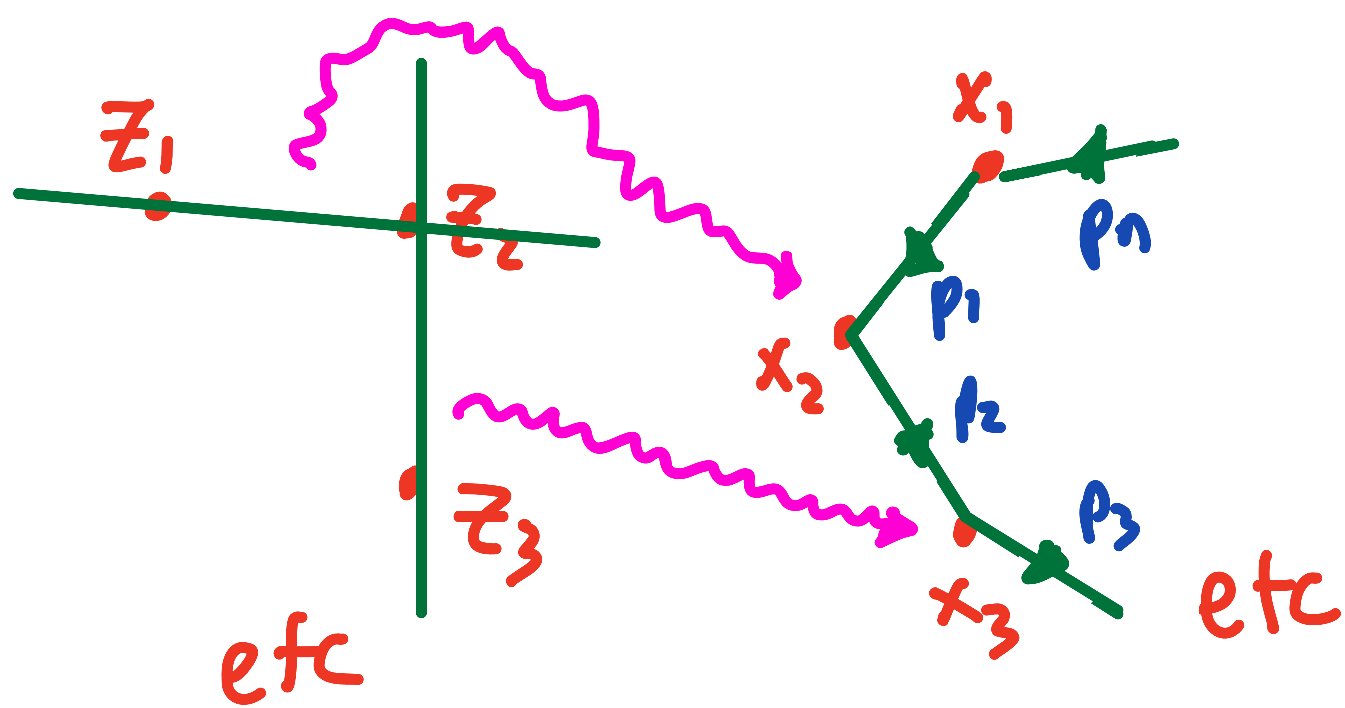


Grassmannians

The takeaway is that the kinematic configuration space is that of n ordered, but otherwise arbitrary, points in $\mathbb{P}^3$.



Summary

a point x in (complexified, compactified) $\mathbb{R}^{1,3}$

$\Longleftrightarrow$ a line $L = \{Z = (\lambda, \mu) : \mu = x\lambda\}$ in $\mathbb{P}^3$

two points x_1, x_2 are null separated in $\mathbb{R}^{1,3}$

$\Longleftrightarrow$ the corresponding lines L_1, L_2 intersect in $\mathbb{P}^3$

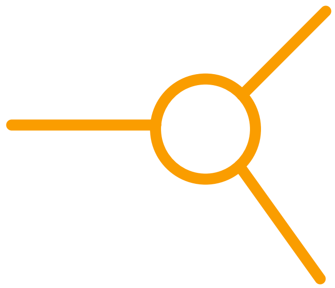
Therefore, a configuration of n ordered points in $\mathbb{P}^3$ $\Longleftrightarrow$ a configuration of n null vectors p_i satisfying $p_1 + p_2 + \cdots + p_n = 0$.

On-Shell Diagrams

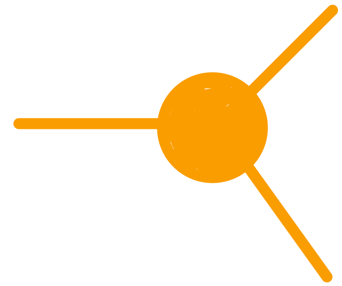
The upshot is that $N^k MHV$ scattering amplitudes of n massless gauge bosons are (at tree-level) rational functions of n ordered homogeneous coordinates on $\mathbb{P}^3$.

They can be computed recursively starting from 2 basic building blocks which henceforth we draw as

On-Shell Diagrams



"anti-MHV" or $\overline{\text{MHV}}$ amplitude



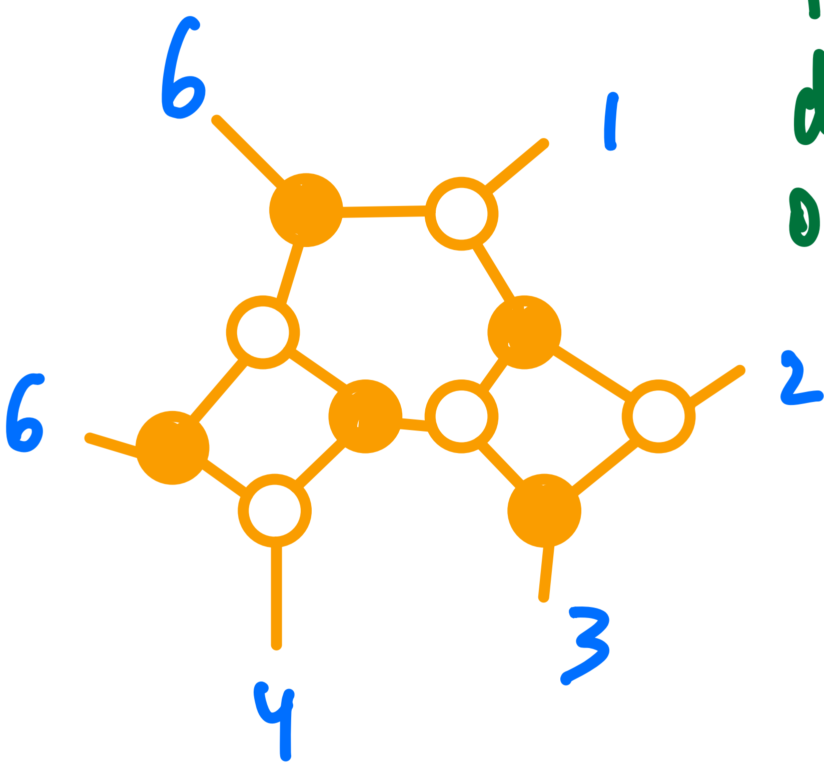
MHV amplitude

$$A(1^+,2^+,3^-) = \frac{[12]^3}{[23][31]}$$

$$A(1^+,2^-,3^-) = \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}$$

On-Shell Diagrams

We can naturally define and compute "generalized" factorizations of n-particle amplitudes



To "compute" an on-shell diagram take the product of indicated MHV and $\overline{\text{MHV}}$ amplitudes, summed over all helicity assignments on the internal edges.

On-Shell Diagrams

Equivalence classes of such plabic graphs under certain graphical moves are in 1-1 correspondence with cells of the non-negative Grassmannian $Gr_{\geq}(k,n)$.

$$\left[\; k = 2\cdot(\#\text{ of MHV vertices}) + 1\cdot(\#\text{ of NMHV vertices}) - (\#\text{ of internal edges}) \;\right]$$

On-Shell Diagrams

Equivalence classes of such plabic graphs under certain graphical moves are in 1-1 correspondence with cells of the non-negative Grassmannian $Gr_{\geq}(k,n)$.

Physics gives us a natural way to associate a rational quantity to each such graph; and those quantities are invariant under the graphical moves.

On-Shell Diagrams

In math, we can also associate a natural rational quantity to each cell of $Gr_{\geq 0}(k,n)$ — its canonical form — defined as the unique differential form that is positive everywhere inside, and has logarithmic singularities on all codimension 1 boundaries, of the cell.

On-Shell Diagrams

Eg if $(\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$ parameterizes the interior of some d dimensional cell (as read off from the plabic graph by a <u>boundary measurement</u>, for example) then the canonical form on the cell is

$$\Omega = \bigwedge_{i=1}^{d} \frac{d\alpha_i}{\alpha_i}$$

On-Shell Diagrams

Eg if $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_+^d$ parameterize the interior of some d dimensional cell (as read off from the plabic graph by a boundary measurement, for example) then the canonical form on the cell is

$$\Omega = \bigwedge_{i=1}^{d} \frac{d\alpha_i}{\alpha_i}$$

This is exactly* the quantity physics computes.

* in maximally supersymmetric gauge theory.

Where is the Amplitude

Plabic graphs/on-shell diagrams give a collection of interesting rational functions with remarkable properties & interrelationships.

Where is the Amplitude

Plabic graphs/on-shell diagrams give a collection of interesting rational functions with remarkable properties & interrelationships.

Natural question: which linear combination of these things computes the tree-level, n-particle, $N^k MHV$ amplitude?

Arkani-Hamed & Trnka conjectured that the answer is given by the amplituhedron.

The Amplituhedron

(see M. Sherman-Bennett)

Let Z be a positive $n \times m$ matrix.

[In physics we need $m=4$, and the rows of Z would be homogeneous coordinates on $\mathbb{P}^3$, but it is easy to generalize on m.]

The Amplituhedron

Let Z be a positive $n \times (k+m)$ matrix.
Then the tree-level amplituhedron $\mathcal{A}_{n,k,m}(Z)$
is the image of the positive Gr. $Gr_+(k,n)$
under the map $C \longmapsto \underbrace{CZ}_{\text{a } k\times(k+m) \text{ matrix}} \in G(k,k+m)$

a $k \times n$ matrix

The Amplituhedron

Let Z be a positive $n \times (k+m)$ matrix.
Then the tree-level amplituhedron $\mathcal{A}_{n,k,m}[Z]$
is the image of the positive Gr. $Gr_+(k,n)$
under the map

$$\underset{\text{a } k\times n \text{ matrix}}{C} \longmapsto \underbrace{C\,Z}_{\text{a } k\times(k+m) \text{ matrix}} \in G(k,k+m)$$

Since $\dim G(k,k+m) = mk \leq \dim G(k,n) = k(n-k)$
for $n \geq k+m$, this is in general a highly
redundant map.

The Amplituhedron

Non-redundant maps into $G(k,k+m)$ can only come from cells of $G_+(k,n)$ that have dimension $m \cdot k$.

The AH-T conjecture is that the canonical form associated to the union (sum) of such cells whose image tiles the amplituhedron non-redundantly <u>is</u> the tree-level amplitude.

Proven by C. Even-Zohar, T. Lakrec, R. Tessler
2112.02703

Part 3: Loops, Cluster Algebras and Tropical Fans

In the following I specialize to planar maximally supersymmetric Yang-Mills theory.

Some References Part 3.

The 2-loop 6-point MHV amplitude

- 1006.5703 Goncharov, MS, Vergu, Volovich

Cluster variables in loop amplitudes

- 1305.1617 Golden, Goncharov, MS, Vergu, Volovich

A review of high loop calculations for $n=6,7$

- 2005.06735 Caron-Huot et al

State of the art amplitude calculations for $n>7$

- 1105.5606 Caron-Huot
- 2009.11471 He, Li, Zhang
- 2110.00350 Li, Zhang

See also the talk by N. Early for some related ideas.

Loop Integrands and Integrals

The L-loop n-particle N^kMHV amplitude is the contribution from the sum over all Feynman diagrams with L closed loops, and hence have L unfixed loop momenta that must be integrated over.

$$\int d^4\ell_1 \cdots d^4\ell_L \underbrace{\left(\text{a rational function of } p_i \text{ and the } \ell\text{'s}\right)}_{\text{this part is called the integrand}}$$

Loop Integrands and Integrals

When we switch to momentum twistor variables, each loop momentum corresponds to a line in twistor space, so the integrand becomes a $4L$-form on the configuration space of L lines in $\mathbb{P}^3$ (represented as $\mathcal{L}_\alpha = A_\alpha B_\alpha$, $\alpha = 1, \dots, L$, L pairs of distinct points in $\mathbb{P}^3$).

Loop Integrands and Integrals

The integrand is moreover a rational function of the "external" kinematic data Z_i and the "internal" loop momentum variables, specifically of things like

$$\langle ijkl \rangle, \quad \langle ij A_\alpha B_\alpha \rangle, \quad \langle A_\alpha B_\alpha A_\beta B_\beta \rangle$$

Loop Integrands and Integrals

To compute the L-loop integral we must integrate this form over some contour in the configuration space of L lines in $\mathbb{P}^3$.

It is not understood what the physical "$+i\varepsilon$" contour is in this space, but actually all possible periods of this form are, in principle, physically and mathematically interesting.

The Loop Amplituhedron $\mathcal{A}_{n,k,m,L}(Z)$

Arkani-Hamed and Trnka also formulated a "loop amplituhedron" in which one considers configurations of lines satisfying certain mutual positivity conditions, and conjectured that the canonical form on the space of such configurations is the loop integrand, but this is much less understood than the tree-level amplituhedron.

The (not quite) simplest nontrivial example of an actual loop amplitude

The $m=4,\ L=2,\ n=6,\ k=0$ amplitude is

$$\mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right)-\frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

+ products of lower-weight Li functions
+ all terms related by $\mathbb{Z}_6$ cyclic

$$\mathrm{Li}_k(z)=\int_0^z\frac{dt}{t}\,\mathrm{Li}_{k-1}(t)\qquad \mathrm{Li}_1(z)=-\log(1-z)$$

A slightly nonobvious surprise

$$\mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\,\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

It is <u>not</u> a surprise that we see a function of Plücker coordinates — it must be a function on the kinematic space

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(\mathbb{C}^*)^{n-1}$$

rescaling the homogeneous coordinates on $\mathbb{P}^3$ that constitute the rows of Z

A slightly nonobvious surprise

What <u>is</u> surprising, if you dig a little deeper, for example at $n=8$, is that while you may see

$$\mathrm{Li}_1\left(\frac{\langle 1256\rangle\langle 2578\rangle(\langle 1237\rangle\langle 4568\rangle-\langle 1238\rangle\langle 4567\rangle)}{\langle 1237\rangle\langle 1258\rangle\langle 2456\rangle\langle 5678\rangle}\right)$$

you will never see

$$\mathrm{Li}_1\left(\frac{\langle 1256\rangle\langle 2578\rangle(\langle 1237\rangle\langle 4568\rangle-\langle 1238\rangle\langle 4567\rangle)}{\langle 1238\rangle\langle 1257\rangle\langle 2456\rangle\langle 5678\rangle}\right)$$

A slightly nonobvious surprise

Maybe some of you immediately see what's "wrong" with the second formula, but I'll need a few more definitions to explain it systematically.

A slightly nonobvious surprise

Let me tantalize you by reminding

- The function $\mathrm{Li}_k(z)$ has branch points at $z=1$ and (for $k>1$) at $z=0$
- Due to unitarity/factorization, the allowed singularities (and monodromies) are very constrained
- At tree level, the location of singularities (poles) is closely tied to geometry.

The argument of the first $\mathrm{Li}_\bullet(\ )$ is valued in $[0,1]$ inside $Gr_+(4,8)$ but the argument of the second $\mathrm{Li}_\bullet(\ )$ can exceed 1 inside $Gr_+(4,8)$.

Generalized Polylogarithms

Let us fix $\mathcal{F}$ = the field of algebraic functions of Plücker coordinates on $Gr(m,n)$.

F_k is a generalized polylogarithm of weight k if there exists a finite set $\{F_{k-1}^{(i)}, \dots\}$ of functions of weight $k-1$ and $R_i \in \mathcal{F}$ such that

$$dF_k = \sum_i F_{k-1}^{(i)} \, d\log R_i$$

seeded by weight-0 = constant functions.

Generalized Polylogarithms

Applying recursively, we can write

$$dF^{(i)}_{k-1} = \sum_j F^{(i,j)}_{k-2} \, d\log R_j \quad \text{etc.}$$

and a symbol alphabet for a generalized polylogarithm function is a choice of basis for the vector space spanned by the (finite) set

$$\{\log R^{(i)}, \log R^{(i,j)}, \ldots, \log R^{(i_1,\ldots,i_k)}\}$$

Generalized Polylogarithms

For example

$\log a \cdot \log b$ $\qquad\qquad$ $\{a, b\}$ $\qquad$ (we suppress writing "log")

$Li_k(z)$, $k>1$ $\qquad\qquad$ $\{z, 1-z\}$

Generalized Polylogarithms

For example

$\log a \cdot \log b$	$\{a, b\}$	(we suppress writing "log")
$\mathrm{Li}_k(z)$, $k > 1$	$\{z, 1-z\}$	

By construction/definition, a generalized polylogarithm function has **branch points** only (and everywhere) on the locus where at least one symbol letter vanishes.

Symbol Alphabets for Amplitudes

For $n<6$ all loop amplitudes are "trivial" (completely, or almost completely, determined)

For $n=6,7$ it is believed – and supported by all evidence available to date – that a symbol alphabet for all k and L is given by

the set of $Gr(4,n)$ cluster variables.

(for $n=6$: 15 Plückers

for $n=7$: 35 Plückers, $\langle 1347\rangle\langle 2567\rangle - \langle 1567\rangle\langle 2347\rangle$

$\langle 1457\rangle\langle 2367\rangle - \langle 1237\rangle\langle 4567\rangle$

Symbol Alphabets for Amplitudes

There is absolutely no understanding, in physics, for why this should be true.

In principle (!!) it should be provable from consideration of

- the loop-level amplituhedron, which determines the 4L-form loop integrands
- the integration contour in the configuation space of lines in $\mathbb{P}^3$

Symbol Alphabets for Amplitudes

It used to be thought that this property is true for arbitrary individual (planar, massless) 6- and 7-particle Feynman diagrams, but we know now that this is false; e.g.

have branch points inside $Gr_+(4,7)$
$\iff$ they have symbol letters that are not cluster variables.

Symbol Alphabets for Amplitudes

It used to be thought that this property is true for arbitrary individual (planar, massless) 6- and 7-particle Feynman diagrams, but we know now that this is false.

However while individual Feynman diagrams may have singularities inside $Gr_+(4,n)$, no known amplitude does; there must be nontrivial cancellations when all Feynman diagrams are summed.

Symbol Alphabets for Amplitudes

However several new features arise for $n \geq 7$:

(i) For $n = 8, 9$ it is expected that all amplitudes are still generalized polylogarithm functions. However, the $Gr(4, n > 7)$ algebras have infinitely many cluster variables. Over 200 (500) of these appear in the symbols of the $n=8$ ($n=9$) amplitudes that have been computed to date.

Symbol Alphabets for Amplitudes

However several new features arise for $n \geq 7$:

(i) However, these calculations are very difficult, and it is not known whether, if we could compute to arbitrarily high loop order L, we would eventually see every cluster variable, or only a finite subset.

(If the latter, it would be very interesting to characterize the subset!)

Symbol Alphabets for Amplitudes

However several new features arise for $n \geq 7$:

(2) It is known that certain algebraic functions of Plücker coordinates — definitely not cluster variables! — appear in symbol alphabets for $n \geq 7$.

Symbol Alphabets for Amplitudes

However several new features arise for $n \geq 7$:

(2) For example, for $n=8$ there is a symbol letter

$$\frac{A - \sqrt{A^2 - 4B}}{A + \sqrt{A^2 + 4B}}$$

$\Uparrow$
$(0,1)$
inside $Gr_+(4,8)$

$$A = \langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle$$

$$B = \langle 1234\rangle\langle 3456\rangle \cdot \langle 5678\rangle\langle 1278\rangle$$

Symbol Alphabets for Amplitudes

However several new features arise for $n \geq 7$:

(2) The algebraic functions that are known to appear are clearly very special, somehow, but we don't yet have a satisfying "theory" to predict what we should or even might encounter in more complicated (higher n, k, L) amplitudes.

Symbol Alphabets for Amplitudes

However several new features arise for $n \geq 7$:

(3) Most confusingly, it is known that $n \geq 10$ particle amplitudes involve functions beyond generalized polylogarithms, having significantly more complicated analytic structure.

A Working Hypothesis (Arkani-Hamed, Lam, NS)

Positive configuration space $\mathrm{Conf}_n^+(\mathbb{P}^{m-1})$ is the image of $\mathrm{Gr}_+(m,n)$ inside the quotient $\mathrm{Conf}_n(\mathbb{P}^{m-1}) = \mathrm{Gr}(m,n)/(\mathbb{C}^*)^{n-1}$

All evidence available to date is consistent with the hypothesis that loop level amplitudes are singular (have branch points) everywhere on the boundary of $\mathrm{Conf}_n^+(\mathbb{P}^3)$ but nowhere inside.

A Working Hypothesis

All evidence available to date is consistent with the hypothesis that loop level amplitudes are singular (have branch points) everywhere on the boundary of $\mathrm{Conf}_n^+(\mathbb{P}^3)$ but nowhere inside.

These words echo the idea of the amplituhedron — that tree amplitudes (or loop integrands) are positive everywhere inside, and have logarithmic singularities on the boundary of some space.

A Working Hypothesis

The problem of thoroughly understanding the analytic structure of amplitudes is therefore closely tied to understanding the boundary of $\mathrm{Conf}_n^+(\mathbb{P}^3)$.

But this is tantamount to identifying a suitable closure or compactification of $\mathrm{Conf}_n^+(\mathbb{P}^3)$, with certain blowups.

Chicken and egg problem: functions $\rightleftarrows$ blowup

A Working Hypothesis

If we knew all amplitudes, they would tell us the effective boundary structure.

Since that's hard, we'll explore various possible geometries (specifically, polytopal realizations of $\mathrm{Conf}_n^+(\mathbb{P}^3)$) and see if they are consistent with the structure of known amplitudes, and what they might imply for ones currently unknown.

Polytopal Realizations of $\overline{\mathrm{Conf}_n^+(\mathbb{P}^3)}$

In 1912.08222 we (also Drummond et al 1912.08217, Henke et al 1912.08254)
we explored candidates that are naturally constructed from "stringy canonical forms" (Arkani-Hamed, He, Lam)

and are dual to (coarsenings of) certain fans associated to the positive tropical Grassmannian. (Speyer, Williams)

Polytopal Realizations of $\overline{\mathrm{Conf}_n^+(\mathbb{P}^3)}$

Let $Z(x_1, \dots, x_d)$ be a parameterization of $\mathrm{Conf}_n^+(\mathbb{P}^3)$ in terms of cluster variables $x_1, \dots, x_d$ of the initial cluster of $Gr(4,n)$.

For a Plücker coordinate P evaluated on Z let $\mathrm{Newt}(P)$ be its Newton polytope in $\mathbb{R}^d$.

Define $\mathcal{C}^+(4,n)$ to be the Minkowski sum of $\mathrm{Newt}(P)$ for all P of the form

$$\langle i, i+1, j, j+1 \rangle \qquad \langle i, j-1, j, j+1 \rangle$$

Polytopal Realizations of $\overline{\mathrm{Conf}_n^+(\mathbb{P}^3)}$

Define $\mathcal{C}^+(4,n)$ to be the Minkowski sum of $\mathrm{Newt}(P)$ for all P of the form

$$\langle i, i+1, j, j+1\rangle \qquad \langle i, j-1, j, j+1\rangle$$

(using <u>all</u> Plückers would give the polytope dual to the Speyer-Williams fan.)

Some physical motivation is that this is the largest subset of Plückers invariant under parity symmetry.

Polytopal Realizations of $\overline{\mathrm{Conf}_n^+(\mathbb{P}^3)}$

For $n=6,7$ the normal rays to the facets of $C^+(4,n)$ are the g-vectors of the $Gr(4,n)$ cluster algebra.

As previously noted, the corresponding cluster variables constitute (it seems) the symbol alphabets for all 6,7 particle amplitudes.

Polytopal Realizations of $\overline{\mathrm{Conf}_n^+(\mathbb{P}^3)}$

For $n=8$, where the cluster algebra is infinite, the polytope $\mathcal{C}^+(4,8)$ has 274 facets.

272 normal rays are g-vectors of $Gr(4,8)$, and all known (non-algebraic) symbol letters are cluster variables corresponding to (a proper subset) of this list.

Maybe the remaining few dozen cluster variables will be found in not-yet-computed amplitudes?

Polytopal Realizations of $\overline{\mathrm{Conf}_n^+(\mathbb{P}^3)}$

For $n=8$, where the cluster algebra is infinite, the polytope $\mathcal{C}^+(4,8)$ has 274 facets.

The remaining 2 normal rays are <u>not</u> g-vectors of $\widetilde{Gr}(4,8)$ — they lie in "gaps" in the union of cluster cones.

[In fact we guessed, and it is now proven, that these 2 rays, and their images under braid transformations (Fraser), complete all the "gaps".]

Algebraic Functions From Polytopes

There is also a natural way in which the 2 "exceptional" facets encode the algebraic symbol letters:

Gr(4,8) is infinite because it contains $A_{1,1}$ subalgebras $\bullet \rightrightarrows \bullet$; if you succesively mutate back & forth you encounter an infinite sequence of cluster variables

$$\left(\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\right)^n + \left(\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\right)^n$$

1912.08217

1912.08254

Algebraic Functions From Polytopes

In 1912.08222 we approached it slightly differently, by using a formula due to Chang, Duan, Fraser, Li 1907.13575 to associate a semistandard Young tableau to each integer point along the exceptional rays. We conjectured (with some evidence) that the associated characters have rational generating functions

$$\sum_{n=0}^{\infty} t^n ch_{\lambda} = \frac{1}{1-At+B^2}$$

poles @ $t = A \pm \sqrt{A^2-4B}$

Summary

We found evidence that some features of 8-particle amplitudes are imprinted in the structure of certain natural polytopal realizations of $\overline{\mathrm{Conf}_8^+(\mathbb{P}^3)}$.

Extended to $n=9$ in

2106.01392 Henke, Papathansiou
2106.01405 Ren, MS, Volovich

Summary and Conclusion

We found evidence that some features of 8-particle amplitudes are imprinted in the structure of certain natural polytopal realizations of $\overline{\mathrm{Conf}_8^+(\mathbb{P}^3)}$.

Summary and Conclusion

We found evidence that some features of 8-particle amplitudes are imprinted in the structure of certain natural polytopal realizations of $\overline{\mathrm{Conf}_8^+(\mathbb{P}^3)}$.

Is there some natural geometric space for which there is a natural mathematical question to which loop amplitudes are the answer?